

Differential Structure of the Hyperbolic Clifford Algebra

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Abstract

This paper presents a thoughtful review of: (a) the Clifford algebra $Cl(H_V)$ of *multivectors* which is naturally associated with a hyperbolic space H_V ; (b) the study of the properties of the duality product of *multivectors* and *multiforms*; (c) the theory of k multivector and l multiform variables multivector extensors over V and (d) the use of the above mentioned structures to present a theory of the parallelism structure on an arbitrary smooth manifold introducing the concepts of covariant derivatives, deformed covariant derivatives and relative covariant derivatives of multivector, multiform fields and extensors fields.

Keywords: Hyperbolic Clifford algebra; multivectors, multiforms and extensor fields; parallelism structure

1 Introduction

In this paper we first present a review of the Hyperbolic Clifford algebra of a real n -dimensional space V . We emphasize that the hyperbolic Clifford algebra is a very important structure for modern theories of Physics, in particular superfields can be seen as ideal sections of an hyperbolic bundle over a spacetime M , see, e.g., [24, 26, 2, 13, 14, 20]. Besides given a thoughtful exposition of the Clifford algebra $Cl(H_V)$ of *multivectors* which is naturally associated with a hyperbolic space H_V , we review also the properties of the *duality product* of multivectors and multiforms and the theory of k multivector and l multiform variables multivector extensors over V . The algebraic theory is then used

to review with enough details the theory of parallelism structures on an arbitrary smooth manifold M , and the concepts of covariant derivatives, deformed derivatives and relative covariant derivatives of multivector, multiform field and extensors fields (all of crucial importance in several physical theories, see, e.g., [12]).

2 Hyperbolic Spaces

2.1 Definition and Basic Properties

Let V, V^* be a pair of dual n -dimensional vector spaces over the real field \mathbb{R} and $V \oplus V^*$ the exterior direct sum of the vector spaces V and V^* . We call hyperbolic structure over V to the pair

$$H_V = (V \oplus V^*, \langle, \rangle)$$

where \langle, \rangle is the non-degenerate symmetric bilinear form of index n defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x^* (y_*) + y^* (x_*) \quad (1)$$

for all $\mathbf{x} = x_* \oplus x^*, \mathbf{y} = y_* \oplus y^* \in H_V$.

Definition 1 *The elements of H_V will be referred as vecfors. A vecfor $\mathbf{x} = x_* \oplus x^* \in H_V$ will be said to be positive if $x^* (x_*) > 0$, null if $x^* (x_*) = 0$ and negative if $x^* (x_*) < 0$. If $x^* (x_*) = 1$ we say that $\mathbf{x} = x_* \oplus x^*$ is a unit vecfor.*

Let H_V^* denote the dual space of H_V , i.e., $H_V^* = ((V \oplus V^*)^*, \langle, \rangle^{-1})$, where \langle, \rangle^{-1} is the reciprocal of the bilinear form \langle, \rangle . We have the following isomorphisms

$$H_V^* \simeq H_{V^*} \simeq H_V$$

where H_{V^*} is the hyperbolic space of V^* . Therefore, H_V is an "auto-dual" space.

The spaces V and V^* are naturally identified with their images in H_V under the inclusions $i_* : V \rightarrow V \oplus V^*, x_* \mapsto i_* x_* = x_* \oplus 0 \equiv x_*$ and $i^* : V^* \rightarrow V \oplus V^*, x^* \mapsto i^* x^* = 0 \oplus x^* \equiv x^*$, then from (1) we have

$$\langle i_* x_*, i_* y_* \rangle = \langle x_*, y_* \rangle = 0, \quad \langle i^* x^*, i^* y^* \rangle = \langle x^*, y^* \rangle = 0, \quad \langle i^* x^*, i_* y_* \rangle = x^* (y_*)$$

for all $x_*, y_* \in V \subset V \oplus V^*$ and $x^*, y^* \in V^* \subset V \oplus V^*$. This means that V and V^* are maximal totally isotropic subspace of H_V and any pair of dual basis satisfying

$$\langle e_i, e_j \rangle = 0, \quad \langle \theta^i, \theta^j \rangle = 0 \quad \text{and} \quad \langle \theta^i, e_j \rangle = \delta_j^i.$$

Remark 2 *More generally, to each subspace $S \subset V$ we can associated a maximal totally isotropic subspace $I(S) \subset H_V$, see [26].*

To each Witt basis $\{e_1, \dots, e_n, \theta^1, \dots, \theta^n\}$ of H_V , we can associated an orthonormal basis $\{\sigma_1, \dots, \sigma_{2n}\}$ of H_V by letting

$$\sigma_k = \frac{1}{\sqrt{2}} (e_k \oplus \theta^k) \quad \text{and} \quad \sigma_{n+k} = \frac{1}{\sqrt{2}} (\bar{e}_k \oplus \theta^k) \quad (2)$$

where $\bar{e}_k = -e_k, k = 1, \dots, n$. We have for $k, l = 1, \dots, n$,

$$\langle \sigma_k, \sigma_l \rangle = \delta_{kl}, \quad \langle \sigma_{n+k}, \sigma_{n+l} \rangle = -\delta_{kl} \quad \text{and} \quad \langle \sigma_k, \sigma_{n+l} \rangle = 0.$$

If $\mathbf{x} = x_* \oplus x^*$, with $x_* = x_*^k e_k$ and $x^* = x_k^* \theta^k$, then the components (x^k, x^{n+k}) of \mathbf{x} with respect to σ_k are

$$x^k = \frac{1}{\sqrt{2}} (x_*^k + x_k^*) \quad \text{and} \quad x^{n+k} = \frac{1}{\sqrt{2}} (x_*^k - x_k^*). \quad (3)$$

Then, base vectors σ_{n+l} are obtained from the σ_k through the involution $\mathbf{x} = x_* \oplus x^* \mapsto \bar{\mathbf{x}} = (-x_*) \oplus x^*$, where $\bar{\mathbf{x}}$ is the (hyperbolic) conjugate of the vector \mathbf{x} .

Due to the auto-duality of H_V stated by the isomorphisms $H_V^* \simeq H_{V^*} \simeq H_V$, we can use the notation $\{\sigma^1, \dots, \sigma^{2n}\}$ to indicate the dual basis $\{\sigma_1, \dots, \sigma_{2n}\}$ as well the reciprocal basis of this same basis. As elements of H_V^* the expressions of the $\sigma^{k'}$ are

$$\sigma^k = \frac{1}{\sqrt{2}} (\theta^k \oplus e_k) \quad \text{and} \quad \sigma_k = \frac{1}{\sqrt{2}} (\bar{\theta}^k \oplus e_k).$$

Another remarkable result on the theory of hyperbolic space is the following (see [17]).

Proposition 3 *Given an arbitrary non-degenerate symmetric bilinear form b on V , there is an isomorphism $H_V \simeq (V, b) \oplus (V, -b) = H_{bV}$.*

2.2 Exterior Algebra of a Hyperbolic Space

The Grassman algebra $\bigwedge H_V$ of the hyperbolic structure H_V is the pair

$$\bigwedge H_V = \left(\bigwedge (V \oplus V^*), \langle, \rangle \right)$$

where

$$\bigwedge (V \oplus V^*) = \sum_{r=0}^{2n} \bigwedge^r (V \oplus V^*)$$

is the exterior algebra of $V \oplus V^*$ and \langle, \rangle is the canonical bilinear form on $\bigwedge (V \oplus V^*)$ induced by the bilinear form \langle, \rangle of H_V , e.g., for simple elements $\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_r, \mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_r \in \bigwedge^r H_V$, \langle, \rangle is given by

$$\langle \mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_r, \mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_r \rangle = \det \begin{pmatrix} \langle \mathbf{x}_1, \mathbf{y}_1 \rangle & \cdots & \langle \mathbf{x}_1, \mathbf{y}_r \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{x}_r, \mathbf{y}_1 \rangle & \cdots & \langle \mathbf{x}_r, \mathbf{y}_r \rangle \end{pmatrix} \quad (4)$$

and it is extended by linearity and orthogonality to all of the algebra $\bigwedge H_V$. Of course, due to the isomorphisms $H_V^* \simeq H_{V^*} \simeq H_V$, we have

$$\bigwedge H_V^* \simeq \bigwedge H_{V^*} \simeq \bigwedge H_V,$$

and it follows that

$$\bigwedge H_V \simeq \left(\bigwedge H_V \right)^*,$$

i.e., $\bigwedge H_V$ is itself an auto-dual space. The elements of $\bigwedge H_V$ will be called *multivecfors*.

Grade involution, reversion and conjugation in the algebra $\bigwedge H_V$ are defined as usual, see [27]. For a homogeneous multivecfors $\mathbf{u} \in \bigwedge^r H_V$,

$$\hat{\mathbf{u}} = (-1)^r \mathbf{u}, \quad \tilde{\mathbf{u}} = (-1)^{\frac{1}{2}r(r-1)} \mathbf{u}, \quad \bar{\mathbf{u}} = (-1)^{\frac{1}{2}r(r+1)} \mathbf{u}$$

and we call that every element $\mathbf{u} \in \bigwedge H_V$ is uniquely decomposed into a sum the even and odd part of \mathbf{u} , i.e., $\mathbf{u} = \mathbf{u}_+ + \mathbf{u}_-$, where

$$\mathbf{u}_+ = \frac{1}{2}(\mathbf{u} + \hat{\mathbf{u}}) \quad \text{and} \quad \mathbf{u}_- = \frac{1}{2}(\mathbf{u} - \hat{\mathbf{u}}).$$

The spaces $\bigwedge V$ and $\bigwedge V^*$ are identified with their images in $\bigwedge H_V$ under the homomorphisms $i^w : \bigwedge V \rightarrow \bigwedge H_V$ defined by

$$i^w(x_{1*} \wedge \dots \wedge x_{r*}) = (x_{1*} \oplus 0) \wedge \dots \wedge (x_{r*} \oplus 0) \equiv x_{1*} \wedge \dots \wedge x_{r*}$$

and $i_w : \bigwedge V^* \rightarrow \bigwedge H_V$ defined by

$$i_w(x_1^* \wedge \dots \wedge x_r^*) = (0 \oplus x_1^*) \wedge \dots \wedge (0 \oplus x_r^*) \equiv x_1^* \wedge \dots \wedge x_r^*.$$

Then, for $u_*, v_* \in \bigwedge V \subset \bigwedge H_V$ and $u^*, v^* \in \bigwedge V^* \subset \bigwedge H_V$, we have

$$\langle u_*, v_* \rangle = 0, \quad \langle u^*, v^* \rangle = 0 \quad \text{and} \quad \langle u^*, v_* \rangle = u^*(v_*). \quad (5)$$

Thus, $\bigwedge V$ and $\bigwedge V^*$ are totally isotropic subspace of $\bigwedge H_V$. But they are no longer maximal, \langle, \rangle being neutral, the dimension of a maximal totally isotropic subspace is 2^{2n-1} , whereas $\dim \bigwedge V = \dim \bigwedge V^* = 2^n$. For elements $\mathbf{u} = u_* \wedge u^*$, $\mathbf{v} = v_* \wedge v^* \in \bigwedge H_V$ with $u_* \in \bigwedge^r V$, $u^* \in \bigwedge^s V^*$, $v_* \in \bigwedge^s V$ and $v^* \in \bigwedge^r V^*$ it holds

$$\langle \mathbf{u}, \mathbf{v} \rangle = (-1)^{rs} u^*(v_*) v^*(u_*).$$

Proposition 4 *There is the following natural isomorphism*

$$\bigwedge H_V \simeq \bigwedge V \hat{\otimes} \bigwedge V^*$$

where $\hat{\otimes}$ denotes the graded tensor product. Moreover, being b a non-degenerate bilinear form on V , it holds also

$$\bigwedge H_V \simeq \bigwedge H_{bV} \hat{\otimes} \bigwedge H_{-bV}.$$

For a proof see, e.g., Greub [15]. The first of the above isomorphisms is given by the mapping $\bigwedge V \widehat{\otimes} \bigwedge V^* \rightarrow \bigwedge H_V$ by

$$u_* \widehat{\otimes} u^* \mapsto i^w u_* \wedge i_w u^*,$$

for all $u_* \in \bigwedge V$ and $u^* \in \bigwedge V^*$. Under this mapping, we can make the identification

$$\bigwedge^r H_V = \sum_{p+q=r} \bigwedge^p V \widehat{\otimes} \bigwedge^q V^*.$$

2.3 Orientation

Besides a canonical metric structure, a hyperbolic space is also provided with a canonical sense of orientation, induced by the $2n$ -vector $\sigma \in \bigwedge^{2n} H_V$ given by

$$\sigma = \sigma_1 \wedge \dots \wedge \sigma_{2n}$$

where $\{\sigma_1, \dots, \sigma_{2n}\}$ is the orthonormal basis of H_V naturally associated with the dual basis $\{e_1, \dots, e_n\}$ of V and $\{\theta^1, \dots, \theta^n\}$ of V^* . Note that from Eq.(4) it follows immediately that

$$\langle \sigma, \sigma \rangle = (-1)^n.$$

The reason for saying that σ define a canonical sense of orientation is that it is independent on the choice of the bases for V . To see this it is enough to verify that

$$\sigma = e_* \wedge \theta^*$$

where

$$e_* = e_1 \wedge \dots \wedge e_n \quad \text{and} \quad \theta^* = \theta^1 \wedge \dots \wedge \theta^n.$$

Thus, under a change of basis in V , e_* transforms as λe_* , with $\lambda \neq 0$, whereas θ^* transforms as $\lambda^{-1} \theta^*$, so that σ remains unchanged.

2.4 Contractions

A left contraction $\lrcorner : \bigwedge H_V \times \bigwedge H_V \rightarrow \bigwedge H_V$ and a right contraction $\llcorner : \bigwedge H_V \times \bigwedge H_V \rightarrow \bigwedge H_V$ are introduced in the algebra $\bigwedge H_V$ in the usual way (see, e.g., [27]), i.e., by

$$\langle \mathbf{u} \lrcorner \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \widetilde{\mathbf{u}} \wedge \mathbf{w} \rangle \quad \text{and} \quad \langle \mathbf{v} \llcorner \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \wedge \widetilde{\mathbf{u}} \rangle,$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \bigwedge H_V$. These operations have the general properties, for all $\mathbf{x}, \mathbf{y} \in H_V$, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \bigwedge H_V$ and $\sigma \in \bigwedge^{2n} H_V$,

$$\begin{aligned}
\mathbf{x} \lrcorner \mathbf{y} &= \mathbf{x} \lrcorner \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle \quad ; \quad 1 \lrcorner \mathbf{u} = \mathbf{u} \lrcorner 1 = \mathbf{u}, \quad \mathbf{x} \lrcorner 1 = 1 \lrcorner \mathbf{x} = 0 \\
(\mathbf{u} \lrcorner \mathbf{v})^\wedge &= \widehat{\mathbf{u}} \lrcorner \widehat{\mathbf{v}}, \quad (\mathbf{u} \lrcorner \mathbf{v})^\wedge = \widehat{\mathbf{u}} \lrcorner \widehat{\mathbf{v}} \quad ; \quad (\mathbf{u} \lrcorner \mathbf{v})^\sim = \widetilde{\mathbf{u}} \lrcorner \widetilde{\mathbf{v}}, \quad (\mathbf{u} \lrcorner \mathbf{v})^\sim = \widetilde{\mathbf{u}} \lrcorner \widetilde{\mathbf{v}} \\
\mathbf{u} \lrcorner (\mathbf{v} \lrcorner \mathbf{w}) &= (\mathbf{u} \wedge \mathbf{v}) \lrcorner \mathbf{w}; \quad (\mathbf{u} \lrcorner \mathbf{v}) \lrcorner \mathbf{w} = \mathbf{u} \lrcorner (\mathbf{v} \wedge \mathbf{w}), \quad (\mathbf{u} \lrcorner \mathbf{v}) \lrcorner \mathbf{w} = \mathbf{u} \lrcorner (\mathbf{v} \lrcorner \mathbf{w}) \\
\mathbf{x} \lrcorner (\mathbf{v} \wedge \mathbf{w}) &= (\mathbf{x} \lrcorner \mathbf{v}) \wedge \mathbf{w} + \widehat{\mathbf{v}} \wedge (\mathbf{x} \lrcorner \mathbf{w}) \quad ; \quad (\mathbf{u} \wedge \mathbf{v}) \lrcorner \mathbf{x} = \mathbf{u} \wedge (\mathbf{v} \lrcorner \mathbf{x}) + (\mathbf{u} \lrcorner \mathbf{x}) \wedge \widehat{\mathbf{v}} \\
\mathbf{x} \wedge (\mathbf{v} \lrcorner \mathbf{w}) &= \widehat{\mathbf{v}} \lrcorner (\mathbf{x} \wedge \mathbf{w}) - (\widehat{\mathbf{v}} \lrcorner \mathbf{x}) \lrcorner \mathbf{w} \quad ; \quad (\mathbf{u} \lrcorner \mathbf{v}) \wedge \mathbf{x} = (\mathbf{u} \wedge \mathbf{x}) \lrcorner \widehat{\mathbf{v}} - \mathbf{u} \lrcorner (\mathbf{x} \lrcorner \widehat{\mathbf{v}}) \\
\mathbf{u}_+ \lrcorner \mathbf{v} &= \mathbf{v} \lrcorner \mathbf{u}_+ \quad ; \quad \mathbf{u}_- \lrcorner \mathbf{v} = \widehat{\mathbf{v}} \lrcorner \widehat{\mathbf{u}}_- \\
\mathbf{u} \wedge (\mathbf{v} \lrcorner \sigma) &= (\mathbf{u} \lrcorner \mathbf{v}) \lrcorner \sigma \quad ; \quad (\sigma \lrcorner \mathbf{v}) \wedge \mathbf{w} = \sigma \lrcorner (\mathbf{v} \lrcorner \mathbf{w})
\end{aligned}$$

Moreover, from Eqs.(5) we get

$$u_* \lrcorner v_* = u_* \lrcorner v_* = 0, \quad u^* \lrcorner v^* = u^* \lrcorner v^* = 0 \quad (6)$$

for all $u_*, v_* \in \bigwedge V$ and $u^*, v^* \in \bigwedge V^*$, so that for elements of the form $\mathbf{u} = u_* \wedge u^*$ and $\mathbf{x} = x_* \oplus x^*$, it holds

$$\begin{aligned}
\mathbf{x} \lrcorner \mathbf{u} &= (x^* \lrcorner u_*) \wedge u^* + \widehat{u}_* \wedge (x_* \lrcorner u^*) \\
\mathbf{u} \lrcorner \mathbf{x} &= u_* \wedge (u^* \lrcorner x_*) + (x^* \lrcorner u_*) \wedge \widehat{u}^*
\end{aligned}$$

For more details about the properties of the left and right contractions, see, e.g., [19, 27].

2.5 Poincaré Automorphism (Hodge Dual)

Define now the Poincaré automorphism or Hodge dual $\star : \bigwedge H_V \rightarrow \bigwedge H_V$ by

$$\star \mathbf{u} = \widetilde{\mathbf{u}} \lrcorner \sigma, \quad (7)$$

for all $\mathbf{u} \in \bigwedge H_V$. The inverse \star^{-1} of this operation is given by

$$\star^{-1} \mathbf{u} = \widetilde{\sigma} \lrcorner \widetilde{\mathbf{u}}. \quad (8)$$

The following general properties of the Hodge duality holds true

$$\begin{aligned}
\star \sigma &= (-1)^n, \quad \star^{-1} \sigma = 1, \quad \langle \star \mathbf{u}, \star \mathbf{v} \rangle = (-1)^n \langle \mathbf{u}, \mathbf{v} \rangle, \\
\star (\mathbf{u} \wedge \mathbf{v}) &= \widetilde{\mathbf{v}} \lrcorner \star \mathbf{u}, \quad \star^{-1} (\mathbf{u} \wedge \mathbf{v}) = (\star^{-1} \mathbf{v}) \lrcorner \widetilde{\mathbf{u}}, \\
\star (\mathbf{u} \lrcorner \mathbf{v}) &= \widetilde{\mathbf{v}} \wedge \star \mathbf{u}, \quad \star^{-1} (\mathbf{u} \lrcorner \mathbf{v}) = (\star^{-1} \mathbf{v}) \wedge \widetilde{\mathbf{u}}.
\end{aligned}$$

For $\mathbf{x} = x_* \oplus x^* \in H_V \subset \bigwedge H_V$ we have, since $x_* \lrcorner e_* = 0$ and $x^* \lrcorner \theta^* = 0$ that

$$\begin{aligned}\star \mathbf{x} &= \mathbf{x} \lrcorner \sigma = \mathbf{x} \lrcorner (e_* \wedge \theta^*) = (\mathbf{x} \lrcorner e_*) \wedge \theta^* + \widehat{e}_* \wedge (\mathbf{x} \lrcorner \theta^*) \\ &= (x^* \lrcorner e_*) \wedge \theta^* - e_* \wedge (\theta^* \lrcorner x_*),\end{aligned}$$

and it follows that, for $u_* \in \bigwedge V \subset \bigwedge H_V$ and $u^* \in \bigwedge V^* \subset \bigwedge H_V$,

$$\star u^* = (\widetilde{u}^* \lrcorner e_*) \wedge \theta^* = D_{\mathfrak{H}} u^* \wedge \theta^* \quad \text{and} \quad \star u_* = e_* \wedge (\theta^* \lrcorner \overline{u}_*) = e_* \wedge D^{\mathfrak{H}} u_*$$

where we introduced the Poincaré isomorphisms $D_{\mathfrak{H}} : \bigwedge V^* \rightarrow \bigwedge V$ and $D^{\mathfrak{H}} : \bigwedge V \rightarrow \bigwedge V^*$ by (see, e.g., [15])

$$D_{\mathfrak{H}} u^* = \widetilde{u}^* \lrcorner e_* \quad \text{and} \quad D^{\mathfrak{H}} u_* = \theta^* \lrcorner \overline{u}_*.$$

For an element of the form $\mathbf{u} = u_* \wedge u^*$ with $u_* \in \bigwedge V$ and $u^* \in \bigwedge V^*$, we have

$$\star \mathbf{u} = D_{\mathfrak{H}} u^* \wedge D^{\mathfrak{H}} u_*.$$

2.6 Clifford Algebra of a Hyperbolic Space

Introduce in $\bigwedge H_V$ the Clifford product of a vector $\mathbf{x} \in H_V$ by an element $\mathbf{u} \in \bigwedge H_V$ by

$$\mathbf{x}\mathbf{u} = \mathbf{x} \lrcorner \mathbf{u} + \mathbf{x} \wedge \mathbf{u}$$

and extend this product by linearity and associativity to all of the space $\bigwedge H_V$. The resulting algebra is isomorphic to the Clifford algebra $\mathcal{C}\ell(H_V)$ of the hyperbolic structure H_V and will thereby be identified with it.

We call $\mathcal{C}\ell(H_V)$ the *mother* algebra (or the hyperbolic Clifford algebra) of the vector space V . The even and odd subspaces of $\mathcal{C}\ell(H_V)$ will be denoted respectively by $\mathcal{C}\ell^{(0)}(H_V)$ and $\mathcal{C}\ell^{(1)}(H_V)$, so that

$$\mathcal{C}\ell(H_V) = \mathcal{C}\ell^{(0)}(H_V) \oplus \mathcal{C}\ell^{(1)}(H_V)$$

and the same notation of the exterior algebra is used for grade involution, reversion, and conjugation in $\mathcal{C}\ell(H_V)$, which obviously satisfy

$$(\mathbf{u}\mathbf{v})^{\wedge} = \hat{\mathbf{u}}\hat{\mathbf{v}}, \quad (\mathbf{u}\mathbf{v})^{\sim} = \tilde{\mathbf{v}}\tilde{\mathbf{u}}, \quad (\mathbf{u}\mathbf{v})^{-} = \bar{\mathbf{v}}\bar{\mathbf{u}}.$$

For vectors $\mathbf{x}, \mathbf{y} \in H_V$, we have the relation

$$\mathbf{x}\mathbf{y} + \mathbf{y}\mathbf{x} = 2 \langle \mathbf{x}, \mathbf{y} \rangle.$$

For the basis elements $\{\sigma_k\}$ it holds

$$\begin{aligned}\sigma_k \sigma_l + \sigma_l \sigma_k &= 2\delta_{kl}, \\ \sigma_{n+k} \sigma_{n+l} + \sigma_{n+l} \sigma_{n+k} &= -2\delta_{kl}, \\ \sigma_k \sigma_{n+l} &= -\sigma_{n+l} \sigma_k.\end{aligned}$$

In turn, for the Witt basis $\{e_k, \theta^k\}$, we have instead

$$\begin{aligned} e_k e_l + e_l e_k &= 0, \\ \theta^k \theta^l + \theta^l \theta^k &= 0, \\ \theta^k e_l + e_l \theta^k &= 2\delta_l^k. \end{aligned}$$

The Clifford product has the following general properties:

$$\begin{aligned} \mathbf{u} \lrcorner \sigma &= \mathbf{u} \sigma, & \sigma \lrcorner \mathbf{u} &= \sigma \mathbf{u}, \\ \langle \mathbf{u}, \mathbf{v} \mathbf{w} \rangle &= \langle \tilde{\mathbf{v}} \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{u} \tilde{\mathbf{w}}, \mathbf{v} \rangle, \\ \mathbf{x} \wedge \mathbf{u} &= \frac{1}{2} (\mathbf{x} \mathbf{u} + \hat{\mathbf{u}} \mathbf{x}), & \mathbf{u} \wedge \mathbf{x} &= \frac{1}{2} (\mathbf{u} \mathbf{x} + \mathbf{x} \hat{\mathbf{u}}), \\ \mathbf{x} \lrcorner \mathbf{u} &= \frac{1}{2} (\mathbf{x} \mathbf{u} - \hat{\mathbf{u}} \mathbf{x}), & \mathbf{u} \lrcorner \mathbf{x} &= \frac{1}{2} (\mathbf{u} \mathbf{x} - \mathbf{x} \hat{\mathbf{u}}), \\ \mathbf{x} \lrcorner (\mathbf{u} \mathbf{v}) &= (\mathbf{x} \lrcorner \mathbf{u}) \mathbf{v} + \hat{\mathbf{u}} (\mathbf{x} \lrcorner \mathbf{v}), & (\mathbf{u} \mathbf{v}) \lrcorner \mathbf{x} &= \mathbf{u} (\mathbf{v} \lrcorner \mathbf{x}) - (\mathbf{u} \lrcorner \mathbf{x}) \hat{\mathbf{v}}, \\ \mathbf{x} \wedge (\mathbf{u} \mathbf{v}) &= (\mathbf{x} \lrcorner \mathbf{u}) \mathbf{v} + \hat{\mathbf{u}} (\mathbf{x} \wedge \mathbf{v}) = (\mathbf{x} \wedge \mathbf{u}) \mathbf{v} - \hat{\mathbf{u}} (\mathbf{x} \lrcorner \mathbf{v}), \\ (\mathbf{u} \mathbf{v}) \wedge \mathbf{x} &= \mathbf{u} (\mathbf{v} \wedge \mathbf{x}) - (\mathbf{u} \lrcorner \mathbf{x}) \hat{\mathbf{v}} = \mathbf{u} (\mathbf{v} \lrcorner \mathbf{x}) + (\mathbf{u} \wedge \mathbf{x}) \hat{\mathbf{v}}, \\ \star \mathbf{u} &= \tilde{\mathbf{u}} \sigma, & \star^{-1} \mathbf{u} &= \tilde{\sigma} \tilde{\mathbf{u}}, \\ \star (\mathbf{u} \mathbf{v}) &= \tilde{\mathbf{v}} (\star \mathbf{u}), & \star^{-1} (\mathbf{u} \mathbf{v}) &= (\star^{-1} \mathbf{v}) \mathbf{u}. \end{aligned}$$

Moreover, for an element of the form $\mathbf{u} = u_* \wedge u^*$, with $u_* \in \bigwedge V$ and $u^* \in \bigwedge V^*$ and $\mathbf{x} \in \bigwedge H_V$ it is:

$$\begin{aligned} \mathbf{x} \mathbf{u} &= (x_* \oplus x^*) \lrcorner (u_* \wedge u^*) + (x_* \oplus x^*) \wedge (u_* \wedge u^*) \\ &= (x^* \lrcorner u_*) \wedge u^* + \hat{u}_* \wedge (x_* \lrcorner u^*) + (x_* \oplus x^*) \wedge (u_* \wedge u^*). \end{aligned}$$

On the other hand,

$$(x_* \oplus x^*) \wedge (u_* \wedge u^*) = \hat{u}_* \wedge (x_* \wedge u^*) + (x^* \wedge u_*) \wedge u^*$$

and then,

$$\begin{aligned} \mathbf{x} \mathbf{u} &= (x^* \lrcorner u_*) \wedge u^* + \hat{u}_* \wedge (x_* \lrcorner u^*) + \hat{u}_* \wedge (x_* \wedge u^*) + (x^* \wedge u_*) \wedge u^* \\ &= ((x^* \lrcorner u_*) + (x^* \wedge u_*)) \wedge u^* + \hat{u}_* \wedge ((x_* \lrcorner u^*) + (x_* \wedge u^*)) \\ &= (x^* u_*) \wedge u^* + \hat{u}_* \wedge (x_* u^*). \end{aligned}$$

Also note that the square of the volume $2n$ -vector σ satisfy

$$\sigma^2 = 1.$$

Proposition 5 *There is the following natural isomorphism*

$$\mathcal{Cl}(H_V) \simeq \text{End}(\bigwedge V).$$

In addition, being b a non-degenerate symmetric bilinear form on V , it holds also

$$\mathcal{Cl}(H_V) \simeq \mathcal{Cl}(H_{bV}) \simeq \mathcal{Cl}(V, b) \hat{\otimes} \mathcal{Cl}(V, -b).$$

Proof. The first isomorphism is given by the extension to $\mathcal{Cl}(H_V)$ of the Clifford map $\varphi : H_V \rightarrow \text{End}(\bigwedge V)$ by $\mathbf{x} \mapsto \varphi_x$, with

$$\varphi_x(u_*) = \frac{1}{\sqrt{2}}(x^* \lrcorner u_* + x_* \wedge u_*),$$

for all $u_* \in \bigwedge V$. The second isomorphism, in turn, is induced from the Clifford map

$$x_* \oplus x^* \mapsto x_+ \widehat{\otimes} 1 + 1 \widehat{\otimes} x_-,$$

with

$$x_{\pm} = \frac{1}{\sqrt{2}}(b^* x^* \pm x_*).$$

Corollary 6 *The even and odd subspaces of the hyperbolic Clifford algebra are*

$$\mathcal{Cl}^{(0)}(H_V) \simeq \text{End}(\bigwedge^{(0)} V) \oplus \text{End}(\bigwedge^{(1)} V)$$

and

$$\mathcal{Cl}^{(1)}(H_V) \simeq L(\bigwedge^{(0)} V, \bigwedge^{(1)} V) \oplus L(\bigwedge^{(1)} V, \bigwedge^{(0)} V)$$

where $L(V, W)$ denotes the space of the linear mappings from V to W and $\bigwedge^{(0)} V$ and $\bigwedge^{(1)} V$ denote respectively the spaces of the even and of odd elements of $\bigwedge V$.

Remark 7 *Recalling the definition of the second order hyperbolic structure, H_V^2 , it follows from proposition 5 that*

$$\mathcal{Cl}(H_V^2) \simeq \text{End}(\mathcal{Cl}(H_V))$$

The algebra $\mathcal{Cl}(H_V^2)$ may be called grandmother algebra of the vector space V .

3 Duality Products of Multivectors and Multiforms, and Extensors

In this section we study the *duality product* of multivectors by multiforms used in the definition of the hyperbolic algebra $\mathcal{Cl}(V \oplus V^*, \langle, \rangle)$ of *multivecfors*. We detail some important properties of the left and right contracted products among the elements of $\bigwedge V$ and $\bigwedge V^*$, by introducing a very useful notation for these products. Next, we give a theory of the *k multivector and l multiform variables multivector (or multiform) extensors* over V (defining the spaces $\text{ext}_k^l(V)$ and $\text{ext}_k^{*l}(V)$) introducing the concept of exterior product of extensors, and of several operators acting on these objects as, e.g., the *adjoint operator*, the *exterior power extension operator* and the *contracted extension operator*. We analyze the properties of these operators with considerable detail.

3.1 Duality Scalar Product of Multivectors and Multiforms

In the Appendix A we briefly recall the exterior algebras of multivectors (elements of $\bigwedge V$) and multiforms (elements of $\bigwedge V^*$) associated with a real vector space V of finite dimension which is need for the following.

Definition 8 *The duality scalar product of a multiform Φ with a multivector X is the scalar $\langle \Phi, X \rangle$ defined by the following axioms:*

For all $\alpha, \beta \in \mathbb{R}$:

$$\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = \alpha\beta. \quad (9)$$

For all $\Phi_p \in \bigwedge^p V^$ and $X^p \in \bigwedge^p V$ (with $1 \leq p \leq n$):*

$$\langle \Phi_p, X^p \rangle = \langle X^p, \Phi_p \rangle = \frac{1}{p!} \Phi_p(e_{j_1}, \dots, e_{j_p}) X^p(\varepsilon^{j_1}, \dots, \varepsilon^{j_p}), \quad (10)$$

where $\{e_j, \varepsilon^j\}$ is any pair of dual bases over V .

For all $\Phi \in \bigwedge V^$ and $X \in \bigwedge V$: if $\Phi = \Phi_0 + \Phi_1 + \dots + \Phi_n$ and $X = X^0 + X^1 + \dots + X^n$, then*

$$\langle \Phi, X \rangle = \langle X, \Phi \rangle = \sum_{p=0}^n \langle \Phi_p, X^p \rangle. \quad (11)$$

We emphasize that the scalar $\Phi_p(e_{j_1}, \dots, e_{j_p}) X^p(\varepsilon^{j_1}, \dots, \varepsilon^{j_p})$ has frame independent character, i.e., it does not depend on the pair of dual bases $\{e_j, \varepsilon^j\}$ used in its evaluation, since Φ_p and X^p are p -linear mappings.

Note that for all $\omega \in V^*$ and $v \in V$ it holds

$$\langle v, \omega \rangle = \langle \omega, v \rangle = \omega(v). \quad (12)$$

We present now two noticeable properties for the duality scalar product between p -forms and p -vectors:

(i) For all $\Phi_p \in \bigwedge^p V^*$, and $v_1, \dots, v_p \in V$:

$$\langle \Phi_p, v_1 \wedge \dots \wedge v_p \rangle = \langle v_1 \wedge \dots \wedge v_p, \Phi_p \rangle = \Phi_p(v_1, \dots, v_p). \quad (13)$$

(ii) For all $\omega^1, \dots, \omega^p \in V^*$ and $v_1, \dots, v_p \in V$:

$$\langle \omega^1 \wedge \dots \wedge \omega^p, v_1 \wedge \dots \wedge v_p \rangle = \det \begin{pmatrix} \omega^1(v_1) & \dots & \omega^1(v_p) \\ \vdots & \ddots & \vdots \\ \omega^p(v_1) & \dots & \omega^p(v_p) \end{pmatrix}. \quad (14)$$

The basic properties for the duality scalar product are the non-degeneracy and the distributive laws on the left and on the right with respect to addition of either multiforms or multivectors, i.e.,

$$\begin{aligned} \langle \Phi, X \rangle &= 0, \text{ for all } \Phi, \text{ then } X = 0, \\ \langle \Phi, X \rangle &= 0, \text{ for all } X, \text{ then } \Phi = 0, \end{aligned} \quad (15)$$

$$\begin{aligned}\langle \Phi + \Psi, X \rangle &= \langle \Phi, X \rangle + \langle \Psi, X \rangle, \\ \langle \Phi, X + Y \rangle &= \langle \Phi, X \rangle + \langle \Phi, Y \rangle.\end{aligned}\tag{16}$$

3.2 Duality Contracted Products of Multivectors and Multiforms

3.2.1 Left Contracted Product

Definition 9 *The left contracted product of a multiform Φ with a multivector X (or, a multivector X with a multiform Φ) is the multivector $\langle \Phi, X |$ (respectively, the multiform $\langle X, \Phi |$) defined by the following axioms:*

For all $\Phi_p \in \bigwedge^p V^$ and $X^p \in \bigwedge^p V$ with $0 \leq p \leq n$:*

$$\langle \Phi_p, X^p | = \langle X^p, \Phi_p | = \langle \tilde{\Phi}_p, X^p \rangle = \langle \Phi_p, \tilde{X}^p \rangle.\tag{17}$$

For all $\Phi_p \in \bigwedge^p V^$ and $X^q \in \bigwedge^q V$ (or $X^p \in \bigwedge^p V$ and $\Phi_q \in \bigwedge^q V^*$) with $0 \leq p < q \leq n$:*

$$\langle \Phi_p, X^q | = \frac{1}{(q-p)!} \langle \tilde{\Phi}_p \wedge \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}}, X^q \rangle e_{j_1} \wedge \dots \wedge e_{j_{q-p}},\tag{18}$$

$$\langle X^p, \Phi_q | = \frac{1}{(q-p)!} \langle \tilde{X}^p \wedge e_{j_1} \wedge \dots \wedge e_{j_{q-p}}, \Phi_q \rangle \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}},\tag{19}$$

where $\{e_j, \varepsilon^j\}$ is any pair of dual bases for V and V^* .

For all $\Phi \in \bigwedge V^*$ and $X \in \bigwedge V$: if $\Phi = \Phi_0 + \Phi_1 + \dots + \Phi_n$ and $X = X^0 + X^1 + \dots + X^n$, then

$$\langle \Phi, X | = \sum_{k=0}^n \sum_{j=0}^{n-k} \langle \Phi_j, X^{k+j} |,\tag{20}$$

$$\langle X, \Phi | = \sum_{k=0}^n \sum_{j=0}^{n-k} \langle X^j, \Phi_{k+j} |.\tag{21}$$

Note that the $(q-p)$ -vector $\langle \Phi_p, X^q |$ and the $(q-p)$ -form $\langle X^p, \Phi_q |$ have frame independent character, i.e., they do not depend on the pair of frames $\{e_j, \varepsilon^j\}$ chosen for calculating them.

The left contracted product has the following basic properties:

Proposition 10 *For all $\Phi_p \in \bigwedge^p V^*$ and $X^q \in \bigwedge^q V$ with $0 \leq p \leq q \leq n$. For all $\Psi_{q-p} \in \bigwedge^{q-p} V^*$, it holds*

$$\langle \langle \Phi_p, X^q |, \Psi_{q-p} \rangle = \langle X^q, \tilde{\Phi}_p \wedge \Psi_{q-p} \rangle.\tag{22}$$

For all $X^p \in \bigwedge^p V$ and $\Phi_q \in \bigwedge^q V^*$ with $0 \leq p \leq q \leq n$. For all $Y^{q-p} \in \bigwedge^{q-p} V$, it holds

$$\langle \langle X^p, \Phi_q |, Y^{q-p} \rangle = \langle \Phi_q, \tilde{X}^p \wedge Y^{q-p} \rangle. \quad (23)$$

For all $X \in \bigwedge V$ and $\Phi, \Psi \in \bigwedge V^*$:

$$\langle \langle \Phi, X |, \Psi \rangle = \langle X, \tilde{\Phi} \wedge \Psi \rangle. \quad (24)$$

For all $\Phi \in \bigwedge V^*$ and $X, Y \in \bigwedge V$:

$$\langle \langle X, \Phi |, Y \rangle = \langle \Phi, \tilde{X} \wedge Y \rangle. \quad (25)$$

Proposition 11 *The left contracted product satisfies the distributive laws on the left and on the right.*

For all $\Phi, \Psi \in \bigwedge V^*$ and $X, Y \in \bigwedge V$:

$$\begin{aligned} \langle (\Phi + \Psi), X | &= \langle \Phi, X | + \langle \Psi, X |, \\ \langle \Phi, (X + Y) | &= \langle \Phi, X | + \langle \Phi, Y |. \end{aligned} \quad (26)$$

For all $X, Y \in \bigwedge V$ and $\Phi, \Psi \in \bigwedge V^*$:

$$\begin{aligned} \langle (X + Y), \Phi | &= \langle X, \Phi | + \langle Y, \Phi |, \\ \langle X, (\Phi + \Psi) | &= \langle X, \Phi | + \langle X, \Psi |. \end{aligned} \quad (27)$$

Proof. We present only the proof of the property given by Eq.(22), the other proofs being somewhat analogous.

First note that if $\Psi_{q-p} \in \bigwedge^{q-p} V^*$ and $\{e_j, \varepsilon^j\}$ is any pair of dual bases for V and V^* , we can write

$$\Psi_{q-p} = \frac{1}{(q-p)!} \langle \Psi_{q-p}, e_{j_1} \wedge \dots \wedge e_{j_{q-p}} \rangle \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}}.$$

Then, using the axiom of the Eq.(18) and the above equation we have

$$\begin{aligned} & \langle \langle \Phi_p, X^q |, \Psi_{q-p} \rangle \\ &= \frac{1}{(q-p)!} \left\langle \left\langle \tilde{\Phi}_p \wedge \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}}, X^q \right\rangle e_{j_1} \wedge \dots \wedge e_{j_{q-p}}, \Psi_{q-p} \right\rangle \\ &= \frac{1}{(q-p)!} \left\langle \tilde{\Phi}_p \wedge \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}}, X^q \right\rangle \langle e_{j_1} \wedge \dots \wedge e_{j_{q-p}}, \Psi_{q-p} \rangle \\ &= \left\langle X^q, \frac{1}{(q-p)!} \langle \Psi_{q-p}, e_{j_1} \wedge \dots \wedge e_{j_{q-p}} \rangle \tilde{\Phi}_p \wedge \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}} \right\rangle \\ &= \left\langle X^q, \tilde{\Phi}_p \wedge \frac{1}{(q-p)!} \langle \Psi_{q-p}, e_{j_1} \wedge \dots \wedge e_{j_{q-p}} \rangle \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}} \right\rangle \\ &= \left\langle X^q, \tilde{\Phi}_p \wedge \Psi_{q-p} \right\rangle, \end{aligned}$$

and the result is proved. ■

3.2.2 Right Contracted Product

Definition 12 *The right contracted product of a multiform Φ with a multivector X (or, a multivector X with a multiform Φ) is the multiform $|\Phi, X\rangle$ (respectively, the multivector $|X, \Phi\rangle$) given by the following axioms:*

For all $\Phi_p \in \bigwedge^p V^$ and $X^p \in \bigwedge^p V$ with $n \geq p \geq 0$:*

$$|\Phi_p, X^p\rangle = |X^p, \Phi_p\rangle = \langle \tilde{\Phi}_p, X^p \rangle = \langle \Phi_p, \tilde{X}^p \rangle. \quad (28)$$

For all $\Phi_p \in \bigwedge^p V^$ and $X^q \in \bigwedge^q V$ (or $X^p \in \bigwedge^p V$ and $\Phi_q \in \bigwedge^q V^*$) with $n \geq p > q \geq 0$:*

$$|\Phi_p, X^q\rangle = \frac{1}{(p-q)!} \langle \Phi_p, e_{j_1} \wedge \cdots \wedge e_{j_{p-q}} \wedge \tilde{X}^q \rangle \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_{p-q}}, \quad (29)$$

$$|X^p, \Phi_q\rangle = \frac{1}{(p-q)!} \langle X^p, \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_{p-q}} \wedge \tilde{\Phi}_q \rangle e_{j_1} \wedge \cdots \wedge e_{j_{p-q}}, \quad (30)$$

where $\{e_j, \varepsilon^j\}$ is any pair of dual bases for V and V^* .

For all $\Phi \in \bigwedge V^*$ and $X \in \bigwedge V$: if $\Phi = \Phi_0 + \Phi_1 + \cdots + \Phi_n$ and $X = X^0 + X^1 + \cdots + X^n$ it is:

$$|\Phi, X\rangle = \sum_{k=0}^n \sum_{j=0}^{n-k} |\Phi_{k+j}, X^j\rangle, \quad (31)$$

$$|X, \Phi\rangle = \sum_{k=0}^n \sum_{j=0}^{n-k} |X^{k+j}, \Phi_j\rangle. \quad (32)$$

The right contracted product satisfies properties similar to the left contracted product (see [4]).

3.3 Duality Adjoint of Extensors

In the Appendix D we briefly recall the extensor concept and the exterior product of extensors associated with a real vector space V of finite dimension which is need for the following. Let $\bigwedge^\diamond V$ be any sum of homogeneous subspaces of $\bigwedge V$. There exist μ integer numbers p_1, \dots, p_μ with $0 \leq p_1 < \cdots < p_\mu \leq n$ such that $\bigwedge^\diamond V = \bigwedge^{p_1} V + \cdots + \bigwedge^{p_\mu} V$. Analogously, if $\bigwedge^\diamond V^*$ is any sum of homogeneous subspaces of $\bigwedge V^*$, then there exist ν integer numbers q_1, \dots, q_ν with $0 \leq q_1 < \cdots < q_\nu \leq n$ such that $\bigwedge^\diamond V^* = \bigwedge^{q_1} V^* + \cdots + \bigwedge^{q_\nu} V^*$.

Definition 13 *The linear mappings*

$$\bigwedge V \ni X \mapsto \langle X \rangle \bigwedge^\diamond V \in \bigwedge V \text{ and } \bigwedge V^* \ni \Phi \mapsto \langle \Phi \rangle \bigwedge^\diamond V^* \in \bigwedge V^*$$

such that if $\bigwedge^\diamond V = \bigwedge^{p_1} V + \cdots + \bigwedge^{p_\mu} V$ and $\bigwedge^\diamond V^ = \bigwedge^{q_1} V^* + \cdots + \bigwedge^{q_\nu} V^*$, then*

$$\langle X \rangle \bigwedge^\diamond V = \langle X \rangle^{p_1} + \cdots + \langle X \rangle^{p_\mu} \text{ and } \langle \Phi \rangle \bigwedge^\diamond V^* = \langle \Phi \rangle_{q_1} + \cdots + \langle \Phi \rangle_{q_\nu} \quad (33)$$

are called the $\bigwedge^\diamond V$ -part operator for multivectors and $\bigwedge^\diamond V^$ -part operator for multiforms, respectively.*

It should be evident that for all $X \in \bigwedge V$ and $\Phi \in \bigwedge V^*$:

$$\langle X \rangle \bigwedge^k V = \langle X \rangle^k, \quad (34)$$

$$\langle \Phi \rangle \bigwedge^k_{V^*} = \langle \Phi \rangle_k. \quad (35)$$

Thus, $\bigwedge^\diamond V$ -part operator and $\bigwedge^\diamond V^*$ -part operator are the generalizations of $\langle \rangle^k$ and $\langle \rangle_k$.

Definition 14 Let τ be a multivector extensor of either one multivector variable or one multiform variable. The duality adjoint of τ is given by:

If $\tau \in \text{ext}(\bigwedge_1^\diamond V; \bigwedge_2^\diamond V)$ (or, $\tau \in \text{ext}(\bigwedge_3^\diamond V^*; \bigwedge_4^\diamond V)$), then $\tau^\Delta \in \text{ext}(\bigwedge_2^\diamond V^*; \bigwedge_1^\diamond V^*)$ (respectively, $\tau^\Delta \in \text{ext}(\bigwedge_4^\diamond V^*; \bigwedge_3^\diamond V)$) defined by

$$\begin{aligned} \tau^\Delta(\Phi) &= \left\langle \Phi, \tau(\langle 1 \rangle \bigwedge_1^\diamond V) \right\rangle \\ &+ \sum_{k=1}^n \frac{1}{k!} \left\langle \Phi, \tau(\langle e_{j_1} \wedge \dots \wedge e_{j_k} \rangle \bigwedge_1^\diamond V) \right\rangle \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_k} \end{aligned} \quad (36)$$

for each $\Phi \in \bigwedge_2^\diamond V^*$ (respectively, for each $\Phi \in \bigwedge_4^\diamond V^*$)

$$\begin{aligned} \tau^\Delta(\Phi) &= \left\langle \Phi, \tau(\langle 1 \rangle \bigwedge_3^\diamond V^*) \right\rangle \\ &+ \sum_{k=1}^n \frac{1}{k!} \left\langle \Phi, \tau(\langle \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_k} \rangle \bigwedge_3^\diamond V^*) \right\rangle e_{j_1} \wedge \dots \wedge e_{j_k} \end{aligned} \quad (37)$$

The basic properties of the adjoint of multivector extensors are given by:

Proposition 15 Let $\tau \in \text{ext}(\bigwedge_1^\diamond V; \bigwedge_2^\diamond V)$. For all $X \in \bigwedge_1^\diamond V$ and $\Phi \in \bigwedge_2^\diamond V^*$, it holds

$$\langle \tau(X), \Phi \rangle = \langle X, \tau^\Delta(\Phi) \rangle. \quad (38)$$

Let $\tau \in \text{ext}(\bigwedge_3^\diamond V^*; \bigwedge_4^\diamond V)$. For all $\Phi \in \bigwedge_3^\diamond V^*$ and $\Psi \in \bigwedge_4^\diamond V^*$, it holds

$$\langle \tau(\Phi), \Psi \rangle = \langle \Phi, \tau^\Delta(\Psi) \rangle. \quad (39)$$

Proof. We present only the proof of the property given by Eq.(38), the other one is similar. First, observe that if $X \in \bigwedge_1^\diamond V$, then there exists μ integer numbers p_1, \dots, p_μ with $0 \leq p_1 < \dots < p_\mu \leq n$ such that $X = X^{p_1} + \dots + X^{p_\mu}$ with $X^{p_i} \in \bigwedge^{p_i} V$, where the $\bigwedge^{p_i} V$ are homogeneous subspace of $\bigwedge V$, thus if $\tau \in \text{ext}(\bigwedge_1^\diamond V; \bigwedge_2^\diamond V)$, we have that

$$\tau : \bigwedge_1^\diamond V \rightarrow \bigwedge_2^\diamond V \quad \text{or} \quad \tau : \bigwedge^{p_1} V + \dots + \bigwedge^{p_\mu} V \rightarrow \bigwedge^{q_1} V + \dots + \bigwedge^{q_\mu} V,$$

where $\bigwedge_1^\diamond V = \bigwedge^{p_1} V + \dots + \bigwedge^{p_\mu} V$ and $\bigwedge_2^\diamond V = \bigwedge^{q_1} V + \dots + \bigwedge^{q_\mu} V$, such that

$$\tau(X) = \tau(X^{p_1} + \dots + X^{p_\mu}) = \tau(X^{p_1}) + \dots + \tau(X^{p_\mu}) \in \bigwedge^{q_1} V + \dots + \bigwedge^{q_\mu} V,$$

thus, we can put

$$\tau(X^{p_i}) \in \bigwedge^{q_i} V \quad \text{or} \quad \tau \Big|_{\bigwedge^{p_i} V} \equiv \tau_{p_i} \in \text{ext}(\bigwedge^{p_i} V; \bigwedge^{q_i} V).$$

Now, with the above observation, we have

$$\begin{aligned} \langle \tau(X), \Phi \rangle &= \langle \tau(X^{p_1} + \dots + X^{p_\mu}), \Phi_{q_1} + \dots + \Phi_{q_\mu} \rangle \\ &= \langle \tau_{p_1}(X^{p_1}) + \dots + \tau_{p_\mu}(X^{p_\mu}), \Phi_{q_1} + \dots + \Phi_{q_\mu} \rangle \\ &= \sum_{i,j} \langle \tau_{p_i}(X^{p_i}), \Phi^{q_j} \rangle, \end{aligned} \quad (40)$$

but, by axiom given in Eq.(11), we have that

$$\langle \tau_{p_i}(X^{p_i}), \Phi^{q_j} \rangle \begin{cases} = 0 & \text{if } p_i \neq q_j \\ \neq 0 & \text{if } p_i = q_j = s_l \end{cases},$$

and from Eq. (40), we can write

$$\langle \tau(X), \Phi \rangle = \langle \tau_{s_1}(X^{s_1}), \Phi^{s_1} \rangle + \dots + \langle \tau_{s_\mu}(X^{s_\mu}), \Phi^{s_\mu} \rangle. \quad (41)$$

Now, if we see $\langle \tau_{s_l}(X^{s_l}), \Phi^{s_l} \rangle$ as a scalar product of s_l -vectors, then we have

$$\langle \tau_{s_l}(X^{s_l}), \Phi^{s_l} \rangle = \langle X^{s_l}, \tau_{s_l}^\Delta \Phi^{s_l} \rangle, \quad (42)$$

and from Eqs. (41), (42), and taking into account Eq. (38) we have

$$\begin{aligned} \langle \tau(X), \Phi \rangle &= \langle \tau_{s_1}(X^{s_1}), \Phi^{s_1} \rangle + \dots + \langle \tau_{s_\mu}(X^{s_\mu}), \Phi^{s_\mu} \rangle \\ &= \langle X^{s_1}, \tau_{s_1}^\Delta \Phi^{s_1} \rangle + \dots + \langle X^{s_\mu}, \tau_{s_\mu}^\Delta \Phi^{s_\mu} \rangle \\ &= \langle X^{s_1} + \dots + X^{s_\mu}, \tau_{s_1}^\Delta \Phi^{s_1} + \dots + \tau_{s_\mu}^\Delta \Phi^{s_\mu} \rangle \\ &= \langle X, \tau^\Delta \Phi \rangle, \end{aligned}$$

and the result is proved. ■

Definition 16 Let σ be a multiform extensor of either one multivector variable or one multiform variable. If $\sigma \in \text{ext}(\bigwedge_1^\diamond V; \bigwedge_2^\diamond V^*)$ (or, $\sigma \in \text{ext}(\bigwedge_3^\diamond V^*; \bigwedge_4^\diamond V^*)$), then The duality adjoint operator $(\cdot)^\Delta$ of $\sigma^\Delta \in \text{ext}(\bigwedge_2^\diamond V; \bigwedge_1^\diamond V^*)$ (respectively, $\sigma^\Delta \in \text{ext}(\bigwedge_4^\diamond V; \bigwedge_3^\diamond V)$) is given by

$$\begin{aligned} \sigma^\Delta(X) &= \left\langle X, \sigma(\langle 1 \rangle \bigwedge_1^\diamond V) \right\rangle \\ &+ \sum_{k=1}^n \frac{1}{k!} \left\langle X, \sigma(\langle e_{j_1} \wedge \dots \wedge e_{j_k} \rangle \bigwedge_1^\diamond V) \right\rangle \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_k} \end{aligned} \quad (43)$$

for each $X \in \bigwedge_2^\diamond V$ (respectively,

$$\begin{aligned} \sigma^\Delta(X) &= \left\langle X, \sigma(\langle 1 \rangle \bigwedge_3^\diamond V^*) \right\rangle \\ &+ \sum_{k=1}^n \frac{1}{k!} \left\langle X, \sigma(\langle \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_k} \rangle \bigwedge_3^\diamond V^*) \right\rangle e_{j_1} \wedge \dots \wedge e_{j_k} \end{aligned} \quad (44)$$

for each $X \in \bigwedge_4^\diamond V$).

The basic properties for the adjoint of multiform extensors are given by:

Let $\sigma \in \text{ext}(\bigwedge_1^\diamond V; \bigwedge_2^\diamond V^*)$. For all $X \in \bigwedge_1^\diamond V$ and $Y \in \bigwedge_2^\diamond V$, it holds

$$\langle \sigma(X), Y \rangle = \langle X, \sigma^\Delta(Y) \rangle. \quad (45)$$

Let $\sigma \in \text{ext}(\bigwedge_3^\diamond V^*; \bigwedge_4^\diamond V^*)$. For all $\Phi \in \bigwedge_3^\diamond V^*$ and $X \in \bigwedge_4^\diamond V$, it holds

$$\langle \sigma(\Phi), X \rangle = \langle \Phi, \sigma^\Delta(X) \rangle. \quad (46)$$

3.4 Extended Operators on Extensors

Let λ be an invertible linear operator on V . As we saw in appendix B and C, λ can be extended by considering two different procedures, non-equivalent, denoted by $\underline{\lambda}$ and $\underline{\gamma}$ both mapping multivectors over V into multivectors over V . However, it is possible to extend the action of $\underline{\lambda}$ and $\underline{\gamma}$ in such a way that they map multivector extensors over V into multivector extensors over V .

3.4.1 Exterior Power Extension Operators on Extensors

Definition 17 *Let λ be an invertible linear operator on V . The exterior power extension of a vector operator λ on multivector extensors is the linear mapping*

$$\text{ext}_k^l(V) \ni \tau \mapsto \underline{\lambda}\tau \in \text{ext}_k^l(V)$$

such that

$$\underline{\lambda}\tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) = \underline{\lambda} \circ \tau(\underline{\lambda}^{-1}(X_1), \dots, \underline{\lambda}^{-1}(X_k), \underline{\lambda}^\Delta(\Phi^1), \dots, \underline{\lambda}^\Delta(\Phi^l)) \quad (47)$$

for each $X_1, \dots, X_k \in \bigwedge V$ and $\Phi^1, \dots, \Phi^l \in \bigwedge V^*$.

It that way $\underline{\lambda}$ can be thought as a linear multivector extensor operator.

For instance, for $\tau \in \text{ext}_1^0(V)$ the above definition means

$$\underline{\lambda}\tau = \underline{\lambda} \circ \tau \circ \underline{\lambda}^{-1}, \quad (48)$$

and for $\tau \in \text{ext}_0^1(V)$ it yields

$$\underline{\lambda}\tau = \underline{\lambda} \circ \tau \circ \underline{\lambda}^\Delta. \quad (49)$$

Let λ be an invertible linear operator on V^* . Analogously to the above case, it is possible to extend the action of $\underline{\lambda}$ in such a way to get a linear operator on $\text{ext}_k^{*l}(V)$. We define

$$\text{ext}_k^{*l}(V) \ni \tau \mapsto \underline{\lambda}\tau \in \text{ext}_k^{*l}(V)$$

such that

$$\underline{\lambda}\tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) = \underline{\lambda} \circ \tau(\underline{\lambda}^\Delta(X_1), \dots, \underline{\lambda}^\Delta(X_k), \underline{\lambda}^{-1}(\Phi^1), \dots, \underline{\lambda}^{-1}(\Phi^l)) \quad (50)$$

for each $X_1, \dots, X_k \in \bigwedge V$ and $\Phi^1, \dots, \Phi^l \in \bigwedge V^*$.

For instance, for $\tau \in \text{ext}_1^0(V)$ the above definition yields

$$\underline{\lambda}\tau = \underline{\lambda} \circ \tau \circ \underline{\lambda}^\Delta, \quad (51)$$

and for $\tau \in \text{ext}_0^1(V)$ it implies that

$$\underline{\lambda}\tau = \underline{\lambda} \circ \tau \circ \underline{\lambda}^{-1}. \quad (52)$$

We give some of the properties of the action of the exterior power extension of a vector operator λ on multivector extensors.

Proposition 18 For all $\tau \in \text{ext}_k^l(V)$ and $\sigma \in \text{ext}_r^s(V)$:

$$\underline{\lambda}(\tau \wedge \sigma) = (\underline{\lambda}\tau) \wedge (\underline{\lambda}\sigma). \quad (53)$$

For all $\tau \in \text{ext}_k^{*l}(V)$ and $\sigma \in \text{ext}_r^s(V)$:

$$\underline{\lambda}\langle \tau, \sigma \rangle = \langle \underline{\lambda}^{-\Delta}\tau, \underline{\lambda}\sigma \rangle, \quad (54)$$

$$\underline{\lambda}\langle \tau, \sigma | = \langle \underline{\lambda}^{-\Delta}\tau, \underline{\lambda}\sigma |, \quad (55)$$

$$\underline{\lambda}|\sigma, \tau \rangle = |\underline{\lambda}\sigma, \underline{\lambda}^{-\Delta}\tau \rangle. \quad (56)$$

We present the proof of the property given by Eq.(55), the other proof are similar.

Proof. Without any loss of generality, we give the proof for the particular case where $\tau \in \text{ext}(\bigwedge_1^\diamond V, \bigwedge_2^\diamond V^*; \bigwedge^\diamond V^*)$ and $\sigma \in \text{ext}(\bigwedge_3^\diamond V, \bigwedge_4^\diamond V^*; \bigwedge^\diamond V)$. Take $X \in \bigwedge_1^\diamond V$, $Y \in \bigwedge_3^\diamond V$ and $\Phi \in \bigwedge_2^\diamond V^*$, $\Psi \in \bigwedge_4^\diamond V^*$. A straightforward calculation, using Eq.(47), Eq.(216), Eq.(188) and Eq.(50), gives

$$\begin{aligned} \underline{\lambda}\langle \tau, \sigma | (X, Y, \Phi, \Psi) &= \underline{\lambda} \circ \langle \tau, \sigma | (\underline{\lambda}^{-1}(X), \underline{\lambda}^{-1}(Y), \underline{\lambda}^\Delta(\Phi), \underline{\lambda}^\Delta(\Psi)) \\ &= \underline{\lambda} \left\langle \tau(\underline{\lambda}^{-1}(X), \underline{\lambda}^\Delta(\Phi)), \sigma(\underline{\lambda}^{-1}(Y), \underline{\lambda}^\Delta(\Psi)) \right\rangle \\ &= \left\langle \underline{\lambda}^{-\Delta} \circ \tau(\underline{\lambda}^{-1}(X), \underline{\lambda}^\Delta(\Phi)), \underline{\lambda} \circ \sigma(\underline{\lambda}^{-1}(Y), \underline{\lambda}^\Delta(\Psi)) \right\rangle \\ &= \left\langle \underline{\lambda}^{-\Delta}\tau(X, \Phi), \underline{\lambda}\sigma(Y, \Psi) \right\rangle \\ &= \left\langle \underline{\lambda}^{-\Delta}\tau, \underline{\lambda}\sigma \right\rangle (X, Y, \Phi, \Psi), \end{aligned}$$

whence, the expected result follows. ■

We present now some of the properties of the action of the exterior power extension of a form operator λ on multi-form extensors.

Proposition 19 For all $\tau \in \text{ext}_k^{*l}(V)$ and $\sigma \in \text{ext}_r^{*s}(V)$:

$$\Delta(\tau \wedge \sigma) = (\Delta\tau) \wedge \Delta(\sigma). \quad (57)$$

For all $\tau \in \text{ext}_k^{*l}(V)$ and $\sigma \in \text{ext}_r^{*s}(V)$:

$$\Delta\langle\tau, \sigma\rangle = \langle\Delta\tau, \Delta^{-\Delta}\sigma\rangle, \quad (58)$$

$$\Delta\langle\tau, \sigma|\rangle = \langle\Delta\tau, \Delta^{-\Delta}\sigma|\rangle, \quad (59)$$

$$\Delta|\sigma, \tau\rangle = |\Delta^{-\Delta}\sigma, \Delta\tau\rangle. \quad (60)$$

The proofs are similar to the one of the previous proposition.

3.4.2 Action of Contracted Extension Operators on Extensors

Definition 20 Let γ be a linear operator on V . Its contracted extension operator is an operator γ which maps map multivector extensors over V into multivector extensors over V . We have

$$\text{ext}_k^l(V) \ni \tau \mapsto \gamma\tau \in \text{ext}_k^l(V)$$

such that

$$\begin{aligned} \gamma\tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) &= \gamma \circ \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) \\ &\quad - \tau(\gamma(X_1), \dots, X_k, \Phi^1, \dots, \Phi^l) \\ &\quad \dots - \tau(X_1, \dots, \gamma(X_k), \Phi^1, \dots, \Phi^l) \\ &\quad + \tau(X_1, \dots, X_k, \gamma^\Delta(\Phi^1), \dots, \Phi^l) \\ &\quad \dots + \tau(X_1, \dots, X_k, \Phi^1, \dots, \gamma^\Delta(\Phi^l)) \end{aligned} \quad (61)$$

for each $X_1, \dots, X_k \in \bigwedge V$ and $\Phi^1, \dots, \Phi^l \in \bigwedge V^*$.

Thus, we can think of γ as a linear *multivector extensor operator*. The definitions of γ for $\bigwedge V$ and $\bigwedge V^*$ are given by Eq.(193) and Eq.(198) respectively.

For instance, for $\tau \in \text{ext}_1^0(V)$ the above definition gives

$$\gamma\tau = \gamma \circ \tau - \tau \circ \gamma = \left[\gamma, \tau \right], \quad (62)$$

and for $\tau \in \text{ext}_0^1(V)$ it yields

$$\gamma\tau = \gamma \circ \tau + \tau \circ \gamma^\Delta. \quad (63)$$

Definition 21 Let γ be an invertible linear operator on V^* . Analogously to the above case, it is possible to generalize the action of γ in such a way to get a linear operator on $ext_k^{*l}(V)$. We have

$$ext_k^{*l}(V) \ni \tau \mapsto \gamma \tau \in ext_k^{*l}(V)$$

such that

$$\begin{aligned} \gamma \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) &= \gamma \circ \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) \\ &\quad + \tau(\gamma^\Delta(X_1), \dots, X_k, \Phi^1, \dots, \Phi^l) \\ &\quad \dots + \tau(X_1, \dots, \gamma^\Delta(X_k), \Phi^1, \dots, \Phi^l) \\ &\quad - \tau(X_1, \dots, X_k, \gamma(\Phi^1), \dots, \Phi^l) \\ &\quad \dots - \tau(X_1, \dots, X_k, \Phi^1, \dots, \gamma(\Phi^l)) \end{aligned} \quad (64)$$

for each $X_1, \dots, X_k \in \bigwedge V$ and $\Phi^1, \dots, \Phi^l \in \bigwedge V^*$.

For instance, for $\tau \in ext_1^{*0}(V)$ the above definition yields

$$\gamma \tau = \gamma \circ \tau + \tau \circ \gamma^\Delta, \quad (65)$$

and for $\tau \in ext_0^{*1}(V)$ it holds

$$\gamma \tau = \gamma \circ \tau - \tau \circ \gamma = [\gamma, \tau]. \quad (66)$$

We present some of the main properties of the action of the contracted extension operator of a vector operator γ on multivector extensors.

Proposition 22 For all $\tau \in ext_k^l(V)$ and $\sigma \in ext_r^s(V)$:

$$\gamma(\tau \wedge \sigma) = (\gamma \tau) \wedge \sigma + \tau \wedge (\gamma \sigma). \quad (67)$$

For all $\tau \in ext_k^{*l}(V)$ and $\sigma \in ext_r^s(V)$:

$$\gamma \langle \tau, \sigma \rangle = - \left\langle \gamma^\Delta \tau, \sigma \right\rangle + \left\langle \tau, \gamma \sigma \right\rangle, \quad (68)$$

$$\gamma \langle \tau, \sigma | = - \left\langle \gamma^\Delta \tau, \sigma \right| + \left\langle \tau, \gamma \sigma \right|, \quad (69)$$

$$\gamma | \sigma, \tau \rangle = \left| \gamma \sigma, \tau \right\rangle - \left| \sigma, \gamma^\Delta \tau \right\rangle. \quad (70)$$

Proof. We present only the proof for the property given by Eq. (67), the others are similar. Without any loss of generality, we give the proof for the particular case where $\tau \in \text{ext}(\bigwedge_1^\diamond V, \bigwedge_2^\diamond V^*; \bigwedge^\diamond V^*)$ and $\sigma \in \text{ext}(\bigwedge_3^\diamond V, \bigwedge_4^\diamond V^*; \bigwedge^\diamond V)$. Take $X \in \bigwedge_1^\diamond V$, $Y \in \bigwedge_3^\diamond V$ and $\Phi \in \bigwedge_2^\diamond V^*$, $\Psi \in \bigwedge_4^\diamond V^*$. By using the definition of γ , we have

$$\begin{aligned} \gamma(\tau \wedge \sigma)(X, Y, \Phi, \Psi) &= \gamma \circ (\tau \wedge \sigma)(X, Y, \Phi, \Psi) - (\tau \wedge \sigma) \left(\gamma X, Y, \Phi, \Psi \right) \\ &\quad - (\tau \wedge \sigma) \left(X, \gamma Y, \Phi, \Psi \right) + (\tau \wedge \sigma) \left(X, Y, \gamma \Phi, \Psi \right) \\ &\quad + (\tau \wedge \sigma) \left(X, Y, \Phi, \gamma \Psi \right). \end{aligned} \tag{71}$$

Now, using the property (197) we can write the first term of right side of the Eq. (71) as

$$\gamma \circ (\tau \wedge \sigma)(X, Y, \Phi, \Psi) = \gamma \circ \tau(X, \Phi) \wedge \sigma(Y, \Psi) + \tau(X, \Phi) \wedge \gamma \circ \sigma(Y, \Psi),$$

and remembering that

$$(\tau \wedge \sigma)(X, Y, \Phi, \Psi) = \tau(X, \Phi) \wedge \sigma(Y, \Psi),$$

the Eq. (71) can be written as

$$\begin{aligned} \gamma(\tau \wedge \sigma)(X, Y, \Phi, \Psi) &= \left[\gamma \circ \tau(X, \Phi) - \tau \left(\gamma X, \Phi \right) + \tau \left(X, \gamma^\Delta \Phi \right) \right] \wedge \sigma(Y, \Psi) \\ &\quad + \tau(X, \Phi) \wedge \left[\gamma \circ \sigma(Y, \Psi) - \sigma \left(\gamma Y, \Psi \right) + \sigma \left(Y, \gamma^\Delta \Psi \right) \right] \end{aligned}$$

or

$$\begin{aligned} \gamma(\tau \wedge \sigma)(X, Y, \Phi, \Psi) &= \left[\left(\gamma \tau \right) (X, \Phi) \right] \wedge \sigma(Y, \Psi) + \tau(X, \Phi) \wedge \left[\left(\gamma \sigma \right) (Y, \Psi) \right] \\ &= \left(\gamma \tau \wedge \sigma + \tau \wedge \gamma \sigma \right) (X, Y, \Phi, \Psi), \end{aligned}$$

and the property is proved. ■

We present some properties for the action of the contacted extension operator of a form operator γ on multiform extensors.

Proposition 23 For all $\tau \in \text{ext}_k^{*l}(V)$ and $\sigma \in \text{ext}_r^{*s}(V)$:

$$\gamma(\tau \wedge \sigma) = (\gamma \tau) \wedge \sigma + \tau \wedge (\gamma \sigma). \tag{72}$$

For all $\tau \in \text{ext}_k^{*l}(V)$ and $\sigma \in \text{ext}_r^s(V)$:

$$\gamma \langle \tau, \sigma \rangle = \left\langle \gamma \tau, \sigma \right\rangle - \left\langle \tau, \gamma^\Delta \sigma \right\rangle, \quad (73)$$

$$\gamma \langle \tau, \sigma | = \left\langle \gamma \tau, \sigma \right| - \left\langle \tau, \gamma^\Delta \sigma \right|, \quad (74)$$

$$\gamma |\sigma, \tau \rangle = - \left| \gamma^\Delta \sigma, \tau \right\rangle + \left| \sigma, \gamma \tau \right\rangle. \quad (75)$$

Proofs are analogous to the ones of the previous proposition.

3.5 Multivector and Multiform Fields

Let U be an open set of a smooth manifold M (i.e., $U \subseteq M$). The set of smooth¹ scalar fields on U , as well-known, has a natural structure of *ring (with identity)*, and it will be denoted by $\mathcal{S}(U)$. The set of smooth vector fields on U , as well-known, have natural structure of *modules over $\mathcal{S}(U)$* . It will be denoted by $\mathcal{V}(U)$. The set of smooth form fields on U *can be identified* with the *dual module* for $\mathcal{V}(U)$. It could be denoted by $\mathcal{V}^*(U)$.

Let M be a n -dimensional differentiable manifold and let $U \subset M$ be an open set. A k -vector mapping

$$X^k : U \longrightarrow \bigcup_{p \in U} \bigwedge^k T_p^* M,$$

such that for each $p \in U$, $X_{(p)}^k \in \bigwedge^k T_p^* M$ is called a *k -vector field on U* .

Such X^k with $1 \leq k \leq n$ is said to be a smooth k -vector field on U , if and only if, for all $\omega^1, \dots, \omega^k \in \mathcal{V}^*(U)$, the scalar mapping defined by

$$U \ni p \mapsto X_{(p)}^k(\omega_{(p)}^1, \dots, \omega_{(p)}^k) \in \mathbb{R} \quad (76)$$

is a smooth scalar field on U .

A multivector mapping

$$X : U \longrightarrow \bigcup_{p \in U} \bigwedge T_p^* M,$$

such that for each $p \in U$, $X_{(p)} \in \bigwedge T_p^* M$ is called a *multivector field on U* .

Any multivector at $p \in M$ can be written as a sum of k -vectors (i.e., homogeneous multivectors of degree k) at $p \in M$, with k running from $k = 0$ to $k = n$, i.e., there exist exactly $n + 1$ homogeneous multivector of degree k fields on U , conveniently denote by X^0, X^1, \dots, X^n , such that for every $p \in U$,

$$X_{(p)} = X_{(p)}^0 + X_{(p)}^1 + \dots + X_{(p)}^n. \quad (77)$$

¹Smooth in this paper means, \mathcal{C}^∞ -differentiable or at least enough differentiable in order for our statements to hold.

We say that X is a smooth multivector field on U when each of one of the X^0, X^1, \dots, X^n is a smooth k -vector field on U .

We emphasize that according with the definitions of smoothness as given above, a smooth k -vector field on U *can be identified* to a k -vector over $\mathcal{V}(U)$, and a smooth multivector field on U *can be seen properly* as a multivector over $\mathcal{V}(U)$. Thus, the set of smooth k -vector fields on U may be denoted by $\bigwedge^k \mathcal{V}(U)$, and the set of smooth multivector fields on U may be denoted by $\bigwedge \mathcal{V}(U)$.

A k -form mapping

$$\Phi_k : U \longrightarrow \bigcup_{p \in U} \bigwedge^k T_p^* M$$

such that for each $p \in U$, $\Phi_{k(p)} \in \bigwedge^k T_p^* M$ is called a *k -form field on U* .

Such Φ_k with $1 \leq k \leq n$ is said to be a smooth k -form field on U , if and only if, for all $v_1, \dots, v_k \in \mathcal{V}(U)$, the scalar mapping defined by

$$U \ni p \mapsto \Phi_{k(p)}(v_1(p), \dots, v_k(p)) \in \mathbb{R} \quad (78)$$

is a smooth scalar field on U .

A multiform mapping

$$\Phi : U \longrightarrow \bigcup_{p \in U} \bigwedge T_p^* M$$

such that for each $p \in U$, $\Phi_{(p)} \in \bigwedge T_p^* M$ is called a *multiform field on U* .

Any multiform at $p \in M$ can be written (see, e.g., [27]) as a sum of k -forms (i.e., homogeneous multiforms of degree k) at $p \in M$ with k running from $k = 0$ to $k = n$. It follows that there exist exactly $n + 1$ homogeneous multiform of degree k fields on U , named as $\Phi_0, \Phi_1, \dots, \Phi_n$ such that

$$\Phi_{(p)} = \Phi_{0(p)} + \Phi_{1(p)} + \dots + \Phi_{n(p)} \quad (79)$$

for every $p \in U$.

We say that Φ is a smooth multiform field on U , if and only if, each of $\Phi_0, \Phi_1, \dots, \Phi_n$ is just a smooth k -form field on U .

Note that according with the definitions of smoothness as given above, a smooth k -form field on U *can be identified* with a k -form over $\mathcal{V}(U)$, and a smooth multiform field on U *can be seen* as a multiform over $\mathcal{V}(U)$. Thus, the set of smooth k -form fields on U will be denoted by $\bigwedge^k \mathcal{V}^*(U)$, and the set of smooth multiform fields on U will be denoted by $\bigwedge \mathcal{V}^*(U)$.

3.5.1 Algebra of Multivector and Multiform Fields

We recall first the module (over a ring) structure operations of the set of smooth multivector fields on U and of the set of smooth multiform fields on U . We recall also the concept of the exterior product both of smooth multivector fields as well as of smooth multiform fields on U and we present the definitions of the duality products of a given smooth multivector fields on U by a smooth multiform field on U .

The addition of multivector fields X and Y , or multiform fields Φ and Ψ , is defined by

$$(X + Y)_{(p)} = X_{(p)} + Y_{(p)}, \quad (80)$$

$$(\Phi + \Psi)_{(p)} = \Phi_{(p)} + \Psi_{(p)}, \quad (81)$$

for every $p \in U$.

The scalar multiplication of a multivector field X , or a multiform field Φ , by a scalar field f , is defined by

$$(fX)_{(p)} = f(p)X_{(p)}, \quad (82)$$

$$(f\Phi)_{(p)} = f(p)\Phi_{(p)}, \quad (83)$$

for every $p \in U$.

The exterior product of multivector fields X and Y , and the exterior product of multiform fields Φ and Ψ , are defined by

$$(X \wedge Y)_{(p)} = X_{(p)} \wedge Y_{(p)}, \quad (84)$$

$$(\Phi \wedge \Psi)_{(p)} = \Phi_{(p)} \wedge \Psi_{(p)}, \quad (85)$$

for every $p \in U$.

Each module, of either the smooth multivector fields on U , or the smooth multiform fields on U , endowed with the respective exterior product has a natural structure of associative algebra. They are called *the exterior algebras of multivector and multiform fields on U* .

The duality scalar product of a multiform field Φ with a multivector field X is (see algebraic details in [4]) the scalar field $\langle \Phi, X \rangle$ defined by

$$\langle \Phi, X \rangle (p) = \langle \Phi_{(p)}, X_{(p)} \rangle, \quad (86)$$

for every $p \in U$.

The duality left contracted product of a multiform field Φ with a multivector field X (or, a multivector field X with a multiform field Φ) is the multivector field $\langle \Phi, X |$ (respectively, the multiform field $\langle X, \Phi |$) defined by

$$\langle \Phi, X |_{(p)} = \langle \Phi_{(p)}, X_{(p)} |, \quad (87)$$

$$\langle X, \Phi |_{(p)} = \langle X_{(p)}, \Phi_{(p)} |, \quad (88)$$

for every $p \in U$.

The duality right contracted product of a multiform field Φ with a multivector field X (or, a multivector field X with a multiform field Φ) is the multiform field $| \Phi, X \rangle$ (respectively, the multivector field $| X, \Phi \rangle$) defined by

$$| \Phi, X \rangle_{(p)} = | \Phi_{(p)}, X_{(p)} \rangle, \quad (89)$$

$$| X, \Phi \rangle_{(p)} = | X_{(p)}, \Phi_{(p)} \rangle, \quad (90)$$

for every $p \in U$.

Each duality contracted product of smooth multivector fields on U with smooth multiform fields on U yields a natural structure of *non-associative* algebra.

3.6 Extensor Fields

Let $T_p M$ the set of all tangent vectors to M at p . A multivector extensor mapping

$$\tau : U \longrightarrow \bigcup_{p \in U} \text{ext}_k^l(T_p M)$$

such that for each $p \in U$, $\tau_{(p)} \in \text{ext}_k^l(T_p M)$ is called a *multivector extensor field of k multivector and l multiform variables on U* .

A multiform extensor mapping

$$v : U \longrightarrow \bigcup_{p \in U} \text{ext}_k^{*l}(T_p M)$$

such that for each $p \in U$, $v_{(p)} \in \text{ext}_k^{*l}(T_p M)$ is called a *multiform extensor field of k multivector and l multiform variables on U* .

In the above formulas, $\text{ext}_k^l(T_p M)$ is a short notation for the space of multivector extensors of k multivector and l multiform variables over $T_p M$, i.e., for each $p \in U$:

$$\text{ext}_k^l(T_p M) := \text{ext}(\bigwedge_1^\diamond T_p M, \dots, \bigwedge_k^\diamond T_p M, \bigwedge_1^\diamond T_p^* M, \dots, \bigwedge_l^\diamond T_p^* M; \bigwedge^\diamond T_p M),$$

and $\text{ext}_k^{*l}(T_p M)$ is a short notation for the space of multiform extensors of k multivector and l multiform variables over $T_p M$, i.e., for each $p \in U$:

$$\text{ext}_k^{*l}(T_p M) := \text{ext}(\bigwedge_1^\diamond T_p M, \dots, \bigwedge_k^\diamond T_p M, \bigwedge_1^\diamond T_p^* M, \dots, \bigwedge_l^\diamond T_p^* M; \bigwedge^\diamond T_p^* M),$$

where $T_p^* M$ is the dual space of $T_p M$.

Let us denote the smooth multivector fields: $U \ni p \mapsto X_{1(p)} \in \bigwedge_1^\diamond T_p M, \dots, U \ni p \mapsto X_{k(p)} \in \bigwedge_k^\diamond T_p M$, and $U \ni p \mapsto X_{(p)} \in \bigwedge^\diamond T_p M$, respectively by $\bigwedge_1^\diamond \mathcal{V}(U), \dots, \bigwedge_k^\diamond \mathcal{V}(U)$, and $\bigwedge^\diamond \mathcal{V}(U)$. Let us denote the smooth multiform fields: $U \ni p \mapsto \Phi_{(p)}^1 \in \bigwedge_1^\diamond T_p^* M, \dots, U \ni p \mapsto \Phi_{(p)}^l \in \bigwedge_l^\diamond T_p^* M$, and $U \ni p \mapsto \Phi_{(p)}^l \in \bigwedge^\diamond T_p^* M$, by $\bigwedge_1^\diamond \mathcal{V}^*(U), \dots, \bigwedge_l^\diamond \mathcal{V}^*(U)$ and $\bigwedge^\diamond \mathcal{V}^*(U)$.

Such a multivector extensor field τ will be said to be smooth, if and only if, for all $X_1 \in \bigwedge_1^\diamond \mathcal{V}(U), \dots, X_k \in \bigwedge_k^\diamond \mathcal{V}(U)$, and for all $\Phi^1 \in \bigwedge_1^\diamond \mathcal{V}^*(U), \dots, \Phi^l \in \bigwedge_l^\diamond \mathcal{V}^*(U)$, the multivector mapping defined by

$$U \ni p \mapsto \tau_{(p)}(X_{1(p)}, \dots, X_{k(p)}, \Phi_{(p)}^1, \dots, \Phi_{(p)}^l) \in \bigwedge^\diamond T_p M \quad (91)$$

is a smooth multivector field on U , (i.e., an object living on $\bigwedge^\diamond \mathcal{V}(U)$).

Such a multiform extensor field v will be said to be smooth, if and only if, for all $X_1 \in \bigwedge_1^\diamond \mathcal{V}(U), \dots, X_k \in \bigwedge_k^\diamond \mathcal{V}(U)$, and for all $\Phi^1 \in \bigwedge_1^\diamond \mathcal{V}^*(U), \dots, \Phi^l \in \bigwedge_l^\diamond \mathcal{V}^*(U)$, the multiform mapping defined by

$$U \ni p \mapsto v_{(p)}(X_{1(p)}, \dots, X_{k(p)}, \Phi_{(p)}^1, \dots, \Phi_{(p)}^l) \in \bigwedge^\diamond T_p^* M \quad (92)$$

is a smooth multiform field on U , (i.e., an object living on $\bigwedge^\diamond \mathcal{V}^*(U)$).

We emphasize² that according with the definitions of smoothness as given above, a smooth $\binom{l}{k}$ multivector extensor field on U *can be identified* to a $\binom{l}{k}$ multivector extensor over $\mathcal{V}(U)$. It is also true that a smooth $\binom{l}{k}$ multiform extensor field on U *can be properly seen* as a $\binom{l}{k}$ multiform extensor over $\mathcal{V}(U)$.

Thus, the set of smooth $\binom{l}{k}$ multivector extensor fields on U is just a module over $\mathcal{S}(U)$ which could be denoted by $ext_k^l \mathcal{V}(U)$. And, the set of smooth $\binom{l}{k}$ multiform extensor fields on U is also a module over $\mathcal{S}(U)$ which can be symbolized as $ext_k^{*l} \mathcal{V}(U)$.

3.6.1 Algebras of Extensor Fields

We define now the exterior products of smooth multivector extensor fields on U and smooth multiform extensor fields on U . We also present the definitions of smooth multivector extensor fields on U with smooth multiform extensor fields on U .

The exterior product of either multivector extensor fields or multiform extensor fields τ and σ is defined as

$$(\tau \wedge \sigma)_{(p)} = \tau_{(p)} \wedge \sigma_{(p)} \quad (93)$$

for every $p \in U$.

Each module over $\mathcal{S}(U)$ of either the smooth multivector extensor fields on U or the smooth multiform extensor fields on U endowed with the respective exterior product is an associative algebra.

The duality scalar product of a multiform extensor field τ with a multivector extensor field σ is a scalar extensor field $\langle \tau, \sigma \rangle$ defined by

$$\langle \tau, \sigma \rangle_{(p)} = \langle \tau_{(p)}, \sigma_{(p)} \rangle, \quad (94)$$

for every $p \in U$.

The duality left contracted product of a multiform extensor field τ with a multivector extensor field σ (or, a multivector extensor field τ with a multiform

²A *short name* for a multivector (or, multiform) extensor of k multivector and l multiform variables could be: a $\binom{l}{k}$ multivector (respectively, multiform) extensor.

extensor field σ) is the multivector extensor field (respectively, the multiform extensor field) denoted by $\langle \tau, \sigma |$ and defined by

$$\langle \tau, \sigma |_{(p)} = \langle \tau_{(p)}, \sigma_{(p)} |, \quad (95)$$

for every $p \in U$.

The duality right contracted product of a multiform extensor field τ with a multivector extensor field σ (or, a multivector extensor field τ with a multiform extensor field σ) is the multiform extensor field (respectively, the multivector extensor field) named as $|\tau, \sigma\rangle$, and defined by

$$|\tau, \sigma\rangle_{(p)} = |\tau_{(p)}, \sigma_{(p)}\rangle \quad (96)$$

for every $p \in U$.

As in the case of multivectors and multiforms, each duality contracted product of smooth multivector extensor fields on U with smooth multiform fields on U yields a non-associative algebra.

4 Parallelism Structure and Covariant Derivatives

In this section we present a theory of a general parallelism structure on an arbitrary real differentiable manifold M of dimension n . We present, with this structure, a detailed theory of the covariant derivatives, deformed covariant derivatives and relative covariant derivatives of *multivector*, *multiform* and *extensor* fields.

We detailed some particular formulas valid for a symmetric parallelism structure. These concepts play an important role in different physical theories, in particular they are essential for those that want to have a deep understanding of the geometric theories of the gravitational field.(see, e.g., [12, 29, 3]) and do not want to mislead the curvature of a connection defined in a manifold M ($\dim M = m$) with the fact that M may be a bended submanifold (a brane) embedded in a Euclidean or pseudo-Euclidean manifold $\tilde{M} \simeq \mathbb{R}^n$ ($n > m$). Bending of a brane is characterized by the shape extensor. See [28] for details.

4.1 Parallelism Structure

Let U be an open set of the smooth manifold M .

Definition 24 *A connection for M is a smooth 2 vector variables vector operator field $U \subset M$,*

$$\Gamma : \mathcal{V}(U) \times \mathcal{V}(U) \longrightarrow \mathcal{V}(U),$$

for all $U \subset M$ such that it satisfies the following axioms:

- (i) *for all $f, g \in \mathcal{S}(U)$ and $a, b, v \in \mathcal{V}(U)$*

$$\Gamma(fa + gb, v) = f\Gamma(a, v) + g\Gamma(b, v), \quad (97)$$

(ii) for all $f, g \in \mathcal{S}(U)$ and $a, v, w \in \mathcal{V}(U)$

$$\Gamma(a, fv + gw) = (af)v + (ag)w + f\Gamma(a, v) + g\Gamma(a, w), \quad (98)$$

The behavior of Γ with respect to the first variable will be called *strong linearity*, and the behavior of Γ with respect to the second variable will be called *quasi linearity*.

The restriction of Γ to U , may be called *parallelism structure* on U . We will denote this by $\langle U, \Gamma \rangle$.

4.2 Covariant Derivative of Multivector and Multiform Fields

Definition 25 The a -Directional Covariant Derivative (a -DCD) of a smooth multivector field on U , associated with $\langle U, \Gamma \rangle$, is the mapping

$$\bigwedge \mathcal{V}(U) \ni X \mapsto \nabla_a X \in \bigwedge \mathcal{V}(U),$$

such that the following axioms are satisfied:

(i) For all $f \in \mathcal{S}(U)$:

$$\nabla_a f = af. \quad (99)$$

(ii) For all $X^k \in \bigwedge^k \mathcal{V}(U)$ with $k \geq 1$:

$$\begin{aligned} \nabla_a X^k(\omega^1, \dots, \omega^k) &= aX^k(\omega^1, \dots, \omega^k) \\ &\quad - X^k(\nabla_a \omega^1, \dots, \omega^k) \dots - X^k(\omega^1, \dots, \nabla_a \omega^k), \end{aligned} \quad (100)$$

for every $\omega^1, \dots, \omega^k \in \mathcal{V}^*(U)$.

(iii) For all $X \in \bigwedge \mathcal{V}(U)$, if $X = \sum_{k=0}^n X^k$, then

$$\nabla_a X = \sum_{k=0}^n \nabla_a X^k. \quad (101)$$

The basic properties of the a -DCD of smooth multivector fields are:

The a -Directional Covariant Derivative Operator (a -DCDO) ∇_a when acting on multivector fields is grade-preserving, i.e.,

$$\text{if } X \in \bigwedge^k \mathcal{V}(U), \text{ then } \nabla_a X \in \bigwedge^k \mathcal{V}(U). \quad (102)$$

For all $f \in \mathcal{S}(U)$, $a, b \in \mathcal{V}(U)$ and $X \in \bigwedge \mathcal{V}(U)$

$$\begin{aligned} \nabla_{a+b} X &= \nabla_a X + \nabla_b X, \\ \nabla_{fa} X &= f \nabla_a X. \end{aligned} \quad (103)$$

For all $f \in \mathcal{S}(U)$, $a \in \mathcal{V}(U)$ and $X, Y \in \bigwedge \mathcal{V}(U)$

$$\begin{aligned}\nabla_a(X + Y) &= \nabla_a X + \nabla_a Y, \\ \nabla_a(fX) &= (af)X + f\nabla_a X.\end{aligned}\tag{104}$$

For all $a \in \mathcal{V}(U)$ and $X, Y \in \bigwedge \mathcal{V}(U)$

$$\nabla_a(X \wedge Y) = (\nabla_a X) \wedge Y + X \wedge \nabla_a Y.\tag{105}$$

Definition 26 The a -DCD of a smooth multiform field on U associated with $\langle U, \Gamma \rangle$ is the mapping

$$\bigwedge \mathcal{V}^*(U) \ni \Phi \mapsto \nabla_a \Phi \in \bigwedge \mathcal{V}^*(U),$$

such that the following axioms are satisfied:

$$(i) \text{ For all } f \in \mathcal{S}(U) : \quad \nabla_a f = af.\tag{106}$$

$$(ii) \text{ For all } \Phi_k \in \bigwedge^k \mathcal{V}^*(U) \text{ with } k \geq 1 :$$

$$\begin{aligned}\nabla_a \Phi_k(v_1, \dots, v_k) &= a\Phi_k(v_1, \dots, v_k) \\ &\quad - \Phi_k(\nabla_a v_1, \dots, v_k) \cdots - \Phi_k(v_1, \dots, \nabla_a v_k),\end{aligned}\tag{107}$$

for every $v_1, \dots, v_k \in \mathcal{V}(U)$.

$$(iii) \text{ For all } \Phi \in \bigwedge \mathcal{V}^*(U) : \text{ if } \Phi = \sum_{k=0}^n \Phi_k, \text{ then}$$

$$\nabla_a \Phi = \sum_{k=0}^n \nabla_a \Phi_k.\tag{108}$$

The basic properties for the a -DCD of smooth multiform fields are:

The a -DCDO ∇_a when acting on multiform fields is grade-preserving, i.e.,

$$\text{if } \Phi \in \bigwedge^k \mathcal{V}^*(U), \text{ then } \nabla_a \Phi \in \bigwedge^k \mathcal{V}^*(U).\tag{109}$$

For all $f \in \mathcal{S}(U)$, $a, b \in \mathcal{V}(U)$ and $\Phi \in \bigwedge \mathcal{V}^*(U)$

$$\begin{aligned}\nabla_{a+b}\Phi &= \nabla_a \Phi + \nabla_b \Phi, \\ \nabla_{fa}\Phi &= f\nabla_a \Phi.\end{aligned}\tag{110}$$

For all $f \in \mathcal{S}(u)$, $a \in \mathcal{V}(U)$ and $\Phi, \Psi \in \bigwedge \mathcal{V}^*(U)$

$$\begin{aligned}\nabla_a(\Phi + \Psi) &= \nabla_a \Phi + \nabla_a \Psi, \\ \nabla_a(f\Phi) &= (af)\Phi + f\nabla_a \Phi.\end{aligned}\tag{111}$$

For all $a \in \mathcal{V}(U)$ and $\Phi, \Psi \in \bigwedge \mathcal{V}^*(U)$

$$\nabla_a(\Phi \wedge \Psi) = (\nabla_a \Phi) \wedge \Psi + \Phi \wedge \nabla_a \Psi. \quad (112)$$

We now present three remarkable properties involving the action of ∇_a on the duality products of multivector and multiform fields.

Proposition 27 *When ∇_a acts on the duality scalar product of $\Phi \in \bigwedge \mathcal{V}^*(U)$ with $X \in \bigwedge \mathcal{V}(U)$ follows the Leibniz rule, i.e.,*

$$a \langle \Phi, X \rangle = \langle \nabla_a \Phi, X \rangle + \langle \Phi, \nabla_a X \rangle. \quad (113)$$

∇_a acting on the duality left contracted product of $\Phi \in \bigwedge \mathcal{V}^*(U)$ with $X \in \bigwedge \mathcal{V}(U)$ (or, $X \in \bigwedge \mathcal{V}(U)$ with $\Phi \in \bigwedge \mathcal{V}^*(U)$) satisfies the Leibniz rule, i.e.,

$$\nabla_a \langle \Phi, X | = \langle \nabla_a \Phi, X | + \langle \Phi, \nabla_a X |, \quad (114)$$

$$\nabla_a \langle X, \Phi | = \langle \nabla_a X, \Phi | + \langle X, \nabla_a \Phi |. \quad (115)$$

∇_a acting on the duality right contracted product of $\Phi \in \bigwedge \mathcal{V}^*(U)$ with $X \in \bigwedge \mathcal{V}(U)$ (or, $X \in \bigwedge \mathcal{V}^*(U)$ with $\Phi \in \bigwedge \mathcal{V}(U)$) satisfies the Leibniz rule, i.e.,

$$\nabla_a |\Phi, X \rangle = |\nabla_a \Phi, X \rangle + |\Phi, \nabla_a X \rangle, \quad (116)$$

$$\nabla_a |X, \Phi \rangle = |\nabla_a X, \Phi \rangle + |X, \nabla_a \Phi \rangle. \quad (117)$$

Proof. We will prove only the statement given by Eq.(114). Take $\Phi \in \bigwedge \mathcal{V}^*(U)$, $X \in \bigwedge \mathcal{V}(U)$ and $\Psi \in \bigwedge \mathcal{V}^*(U)$. By a property of the duality left contracted product, we have

$$\langle \langle \Phi, X |, \Psi \rangle = \langle X, \tilde{\Phi} \wedge \Psi \rangle. \quad (a)$$

Now, by using Eq.(113) and Eq.(112), we can write

$$\begin{aligned} & \langle \nabla_a \langle \Phi, X |, \Psi \rangle + \langle \langle \Phi, X |, \nabla_a \Psi \rangle \\ &= \langle \nabla_a X, \tilde{\Phi} \wedge \Psi \rangle + \langle X, \nabla_a (\tilde{\Phi} \wedge \Psi) \rangle \\ &= \langle \nabla_a X, \tilde{\Phi} \wedge \Psi \rangle + \langle X, (\nabla_a \tilde{\Phi}) \wedge \Psi \rangle + \langle X, \tilde{\Phi} \wedge \nabla_a \Psi \rangle. \end{aligned} \quad (b)$$

Thus, taking into account Eq.(109) and by recalling once again a property of the duality left contracted product, it follows that

$$\langle \nabla_a \langle \Phi, X |, \Psi \rangle = \langle \langle \Phi, \nabla_a X |, \Psi \rangle + \langle \langle \nabla_a \Phi, X |, \Psi \rangle \rangle. \quad (c)$$

Then, by the non-degeneracy of the duality scalar product, the required result follows. ■

4.3 Covariant Derivative of Extensor Fields

Definition 28 Let $\langle U, \Gamma \rangle$ be a parallelism structure on U , and let us take $a \in \mathcal{V}(U)$. The a -DCD, associated with $\langle U, \Gamma \rangle$, of a smooth multivector extensor field on U or a smooth multiform extensor field on U are the mappings

$$ext_k^l \mathcal{V}(U) \ni \tau \mapsto \nabla_a \tau \in ext_k^l \mathcal{V}(U),$$

and

$$*^l_k ext_k^l \mathcal{V}(U) \ni \tau \mapsto \nabla_a \tau \in *^l_k ext_k^l \mathcal{V}(U),$$

such that for all $X_1 \in \bigwedge_1^\diamond \mathcal{V}(U), \dots, X_k \in \bigwedge_k^\diamond \mathcal{V}(U)$, and for all $\Phi^1 \in \bigwedge_1^\diamond \mathcal{V}^*(U), \dots, \Phi^l \in \bigwedge_l^\diamond \mathcal{V}^*(U)$ we have

$$\begin{aligned} \nabla_a \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) &= \nabla_a (\tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l)) \\ &\quad - \tau(\nabla_a X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) - \dots \\ &\quad - \tau(X_1, \dots, \nabla_a X_k, \Phi^1, \dots, \Phi^l) \\ &\quad - \tau(X_1, \dots, X_k, \nabla_a \Phi^1, \dots, \Phi^l) - \dots \\ &\quad - \tau(X_1, \dots, X_k, \Phi^1, \dots, \nabla_a \Phi^l). \end{aligned} \quad (118)$$

The covariant derivative of smooth multivector (or multiform) extensor fields has two basic properties:

- (i) For $f \in \mathcal{S}(U)$, and $a, b \in \mathcal{V}(U)$, and $\tau \in ext_k^l \mathcal{V}(U)$ (or $\tau \in *^l_k ext_k^l \mathcal{V}(U)$)

$$\nabla_{a+b} \tau = \nabla_a \tau + \nabla_b \tau \quad (119)$$

$$\nabla_{fa} \tau = f \nabla_a \tau. \quad (120)$$

- (ii) For $f \in \mathcal{S}(U)$, and $a \in \mathcal{V}(U)$, and $\tau, \sigma \in ext_k^l \mathcal{V}(U)$ (or $\tau, \sigma \in *^l_k ext_k^l \mathcal{V}(U)$)

$$\nabla_a (\tau + \sigma) = \nabla_a \tau + \nabla_a \sigma, \quad (121)$$

$$\nabla_a (f\tau) = (af)\tau + f\nabla_a \tau. \quad (122)$$

The covariant differentiation of the exterior product of smooth multivector (or multiform) extensor fields satisfies the Leibniz's rule. We have

Proposition 29 For all $\tau \in ext_k^l \mathcal{V}(U)$ and $\sigma \in ext_r^s \mathcal{V}(U)$ (or, $\tau \in *^l_k ext_k^l \mathcal{V}(U)$ and $\sigma \in *^s_r ext_r^s \mathcal{V}(U)$), it holds

$$\nabla_a (\tau \wedge \sigma) = (\nabla_a \tau) \wedge \sigma + \tau \wedge (\nabla_a \sigma). \quad (123)$$

Proof. Without loss of generality, we prove this statement only for multivector extensor fields $(X, \Phi) \mapsto \tau(X, \Phi)$ and $(Y, \Psi) \mapsto \sigma(Y, \Psi)$. Using Eq.(118), we can

write

$$\begin{aligned}
& \nabla_a(\tau \wedge \sigma)(X, Y, \Phi, \Psi) \\
&= \nabla_a((\tau \wedge \sigma)(X, Y, \Phi, \Psi)) \\
&- (\tau \wedge \sigma)(\nabla_a X, Y, \Phi, \Psi) - (\tau \wedge \sigma)(X, \nabla_a Y, \Phi, \Psi) \\
&- (\tau \wedge \sigma)(X, Y, \nabla_a \Phi, \Psi) - (\tau \wedge \sigma)(X, Y, \Phi, \nabla_a \Psi).
\end{aligned}$$

Using Eq.(93) and recalling Leibniz's rule for the covariant differentiation of the exterior product of multivector fields, we have

$$\begin{aligned}
& \nabla_a(\tau \wedge \sigma)(X, Y, \Phi, \Psi) \\
&= \nabla_a(\tau(X, \Phi)) \wedge \sigma(Y, \Psi) + \tau(X, \Phi) \wedge \nabla_a(\sigma(Y, \Psi)) \\
&- \tau(\nabla_a X, \Phi) \wedge \sigma(Y, \Psi) - \tau(X, \Phi) \wedge \sigma(\nabla_a Y, \Psi) \\
&- \tau(X, \nabla_a \Phi) \wedge \sigma(Y, \Psi) - \tau(X, \Phi) \wedge \sigma(Y, \nabla_a \Psi),
\end{aligned}$$

i.e.,

$$\begin{aligned}
& \nabla_a(\tau \wedge \sigma)(X, Y, \Phi, \Psi) \\
&= (\nabla_a(\tau(X, \Phi)) - \tau(\nabla_a X, \Phi) - \tau(X, \nabla_a \Phi)) \wedge \sigma(Y, \Psi) \\
&+ \tau(X, \Phi) \wedge (\nabla_a(\sigma(Y, \Psi)) - \sigma(\nabla_a Y, \Psi) - \sigma(Y, \nabla_a \Psi)).
\end{aligned}$$

Then, using once again Eq.(118) and Eq.(93), the expected result follows. ■

The covariant differentiation of the duality scalar product and each one of the duality contracted products of smooth extensor fields satisfies the Leibniz's rule.

Proposition 30 For all $\tau \in \text{ext}_k^{*l} \mathcal{V}(U)$ and $\sigma \in \text{ext}_r^s \mathcal{V}(U)$ (or, $\tau \in \text{ext}_k^l \mathcal{V}(U)$ and $\sigma \in \text{ext}_r^{*s} \mathcal{V}(U)$), we have that

$$\nabla_a \langle \tau, \sigma \rangle = \langle \nabla_a \tau, \sigma \rangle + \langle \tau, \nabla_a \sigma \rangle. \quad (124)$$

For all $\tau \in \text{ext}_k^{*l} \mathcal{V}(U)$ and $\sigma \in \text{ext}_r^s \mathcal{V}(U)$ (or, $\tau \in \text{ext}_k^l \mathcal{V}(U)$ and $\sigma \in \text{ext}_r^{*s} \mathcal{V}(U)$), it holds

$$\nabla_a \langle \tau, \sigma | = \langle \nabla_a \tau, \sigma | + \langle \tau, \nabla_a \sigma |, \quad (125)$$

$$\nabla_a \langle \sigma, \tau | = \langle \nabla_a \sigma, \tau | + \langle \sigma, \nabla_a \tau |. \quad (126)$$

For all $\tau \in \text{ext}_k^{*l} \mathcal{V}(U)$ and $\sigma \in \text{ext}_r^s \mathcal{V}(U)$ (or, $\tau \in \text{ext}_k^l \mathcal{V}(U)$ and $\sigma \in \text{ext}_r^{*s} \mathcal{V}(U)$), it holds

$$\nabla_a |\tau, \sigma \rangle = |\nabla_a \tau, \sigma \rangle + |\tau, \nabla_a \sigma \rangle, \quad (127)$$

$$\nabla_a |\sigma, \tau \rangle = |\nabla_a \sigma, \tau \rangle + |\sigma, \nabla_a \tau \rangle. \quad (128)$$

Proof. We present only the proof of the property given by Eq.(125). Without loss of generality, we prove this statement only for a multiform extensor field τ and a multivector extensor field σ such that $(X, \Phi) \mapsto \tau(X, \Phi)$ and $(Y, \Psi) \mapsto \sigma(Y, \Psi)$.

$$\begin{aligned} \nabla_a \langle \tau, \sigma | (X, Y, \Phi, \Psi) &= \nabla_a (\langle \tau, \sigma | (X, Y, \Phi, \Psi)) - \langle \tau, \sigma | (\nabla_a X, Y, \Phi, \Psi) \\ &\quad - \langle \tau, \sigma | (X, \nabla_a Y, \Phi, \Psi) - \langle \tau, \sigma | (X, Y, \nabla_a \Phi, \Psi) \\ &\quad - \langle \tau, \sigma | (X, Y, \Phi, \nabla_a \Psi) \end{aligned}$$

or recalling that $\langle \tau, \sigma |_{(p)} = \langle \tau_{(p)}, \sigma_{(p)} |$,

$$\begin{aligned} \nabla_a \langle \tau, \sigma | (X, Y, \Phi, \Psi) &= \nabla_a (\langle \tau(X, \Phi), \sigma(Y, \Psi) |) - \langle \tau(\nabla_a X, \Phi), \sigma(Y, \Psi) | \\ &\quad - \langle \tau(X, \Phi), \sigma(\nabla_a Y, \Psi) | - \langle \tau(X, \nabla_a \Phi), \sigma(Y, \Psi) | \\ &\quad - \langle \tau(X, \Phi), \sigma(Y, \nabla_a \Psi) | . \end{aligned} \tag{129}$$

On the other hand, from Eq.(116), we can write

$$\nabla_a (\langle \tau(X, \Phi), \sigma(Y, \Psi) |) = \langle \nabla_a \tau(X, \Phi), \sigma(Y, \Psi) | + \langle \tau(X, \Phi), \nabla_a \sigma(Y, \Psi) | ,$$

Eq.(129) can be written as

$$\begin{aligned} \nabla_a \langle \tau, \sigma | (X, Y, \Phi, \Psi) &= \langle \nabla_a \tau(X, \Phi) - \tau(\nabla_a X, \Phi) - \tau(X, \nabla_a \Phi), \sigma(Y, \Psi) | \\ &\quad + \langle \tau(X, \Phi), \nabla_a \sigma(Y, \Psi) - \sigma(\nabla_a Y, \Psi) - \sigma(Y, \nabla_a \Psi) | \\ &= \langle (\nabla_a \tau)(X, \Phi), \sigma(Y, \Psi) | + \langle \tau(X, \Phi), (\nabla_a \sigma)(Y, \Psi) | \\ &= \langle \nabla_a \tau, \sigma | (X, Y, \Phi, \Psi) + \langle \tau, \nabla_a \sigma | (X, Y, \Phi, \Psi) , \end{aligned}$$

which proves our result. ■

Finally we prove that the duality adjoint operator commutes with the a -DCDO, i.e., we have

Proposition 31 *If τ is any one of the four smooth one-variable extensor fields on U , then*

$$(\nabla_a \tau)^\Delta = \nabla_a \tau^\Delta. \tag{130}$$

Proof. Without loss of generality, we prove this statement only for $\tau \in \text{ext}(\bigwedge_1^\diamond \mathcal{V}(U), \bigwedge^\diamond \mathcal{V}(U))$.

Let us $X \in \bigwedge_1^\diamond \mathcal{V}(U)$ and $\Phi \in \bigwedge^\diamond \mathcal{V}^*(U)$. We must prove that

$$\langle \nabla_a \tau^\Delta(\Phi), X \rangle = \langle \Phi, \nabla_a \tau(X) \rangle .$$

By using Eq.(118) and recalling the Leibniz's rule for the covariant differentiation of the duality scalar product of multiform fields with multivector fields, we can write

$$\langle \nabla_a \tau^\Delta(\Phi), X \rangle = \langle \nabla_a (\tau^\Delta(\Phi)), X \rangle - \langle \tau^\Delta(\nabla_a(\Phi)), X \rangle ,$$

and from Eq.(113)

$$\langle \nabla_a \tau^\Delta(\Phi), X \rangle = a \langle \tau^\Delta(\Phi), X \rangle - \langle \tau^\Delta(\Phi), \nabla_a X \rangle - \langle \tau^\Delta(\nabla_a \Phi), X \rangle.$$

Recalling now the fundamental property of the duality *adjoint*, we get

$$\langle \nabla_a \tau^\Delta(\Phi), X \rangle = a \langle \Phi, \tau(X) \rangle - \langle \Phi, \tau(\nabla_a X) \rangle - \langle \nabla_a \Phi, \tau(X) \rangle.$$

Using once again the Leibniz's rule, Eq.(113), for the covariant differentiation of the duality scalar product we get

$$\langle \nabla_a \tau^\Delta(\Phi), X \rangle = \langle \Phi, \nabla_a(\tau(X)) \rangle - \langle \Phi, \tau(\nabla_a X) \rangle,$$

and thus, using once again Eq.(118), the required result follows. ■

In particular, the *a-DCD*, of a smooth *k-covariant and l-contravariant vector (or form) extensor field* on *U*, and associated with $\langle U, \Gamma \rangle$, are defined by

$$ext_k^l \mathcal{V}(U) \ni \tau \mapsto \nabla_a \tau \in ext_k^l \mathcal{V}(U) \text{ and } ext_k^{*l} \mathcal{V}(U) \ni \tau \mapsto \nabla_a \tau \in ext_k^{*l} \mathcal{V}(U),$$

respectively, such that for every $v_1, \dots, v_k \in \mathcal{V}(U)$, and $\omega^1, \dots, \omega^l \in \mathcal{V}^*(U)$:

$$\begin{aligned} \nabla_a \tau(v_1, \dots, v_k, \omega^1, \dots, \omega^l) &= \nabla_a(\tau(v_1, \dots, v_k, \omega^1, \dots, \omega^l)) \\ &\quad - \tau(\nabla_a v_1, \dots, v_k, \omega^1, \dots, \omega^l) - \dots \\ &\quad - \tau(v_1, \dots, \nabla_a v_k, \omega^1, \dots, \omega^l) \\ &\quad - \tau(v_1, \dots, v_k, \nabla_a \omega^1, \dots, \omega^l) - \dots \\ &\quad - \tau(v_1, \dots, v_k, \omega^1, \dots, \nabla_a \omega^l). \end{aligned} \quad (131)$$

Where we can easily see that it meets the basic properties (119), (120), (121) and (122).

It is also worth recalling here that the *a-DCD* of a smooth vector field on *U*, associated with $\langle U, \Gamma \rangle$, is the mapping

$$\mathcal{V}(U) \ni v \mapsto \nabla_a v \in \mathcal{V}(U)$$

such that

$$\nabla_a v = \Gamma(a, v), \quad (132)$$

and the *a-DCD* of a smooth form field on *U*, is the mapping

$$\mathcal{V}^*(U) \ni \omega \mapsto \nabla_a \omega \in \mathcal{V}^*(U)$$

such that for every $v \in \mathcal{V}(U)$

$$\nabla_a \omega(v) = a\omega(v) - \omega(\nabla_a v). \quad (133)$$

4.4 Deformed Covariant Derivative

Let $\langle U, \Gamma \rangle$ be a parallelism structure on U . Let us take an invertible smooth extensor operator field λ on $V \supseteq U$, i.e., $\lambda : \mathcal{V}(U) \rightarrow \mathcal{V}(U)$. The λ -deformation of $\langle U, \Gamma \rangle$ is a well-defined connection on U , namely $\overset{\lambda}{\Gamma}$, given by

$$\mathcal{V}(U) \times \mathcal{V}(U) \ni (a, v) \mapsto \overset{\lambda}{\Gamma}(a, v) \in \mathcal{V}(U)$$

such that

$$\overset{\lambda}{\Gamma}(a, v) = \lambda(\Gamma(a, \lambda^{-1}(v))). \quad (134)$$

It is easy to see that $\overset{\lambda}{\Gamma}$ is indeed a connection on U since it satisfies Eq.(97) and Eq.(98).

The parallelism structure $\langle U, \overset{\lambda}{\Gamma} \rangle$ is said to be the λ -deformation of $\langle U, \Gamma \rangle$.

Let us take $a \in \mathcal{V}(U)$. The a -DCDO associated with $\langle U, \overset{\lambda}{\Gamma} \rangle$, namely $\overset{\lambda}{\nabla}_a$, has the basic properties:

Proposition 32 For all $v \in \mathcal{V}(U)$

$$\overset{\lambda}{\nabla}_a v = \lambda(\nabla_a \lambda^{-1}(v)). \quad (135)$$

It follows from Eq.(132) and Eq.(134).

For all $\omega \in \mathcal{V}^*(U)$

$$\overset{\lambda}{\nabla}_a \omega = \lambda^{-\Delta}(\nabla_a \lambda^\Delta(\omega)). \quad (136)$$

Proof. Eq.(135 follows from Eq.(132) and Eq.(134)). To prove Eq.(136) take $v \in \mathcal{V}(U)$. Then using Eq.(133) and Eq.(135), we have

$$\begin{aligned} \overset{\lambda}{\nabla}_a \omega(v) &= a\omega(v) - \omega(\overset{\lambda}{\nabla}_a v) \\ &= a\omega(v) - \omega(\lambda(\nabla_a \lambda^{-1}(v))) \\ &= a \langle \omega, v \rangle - \langle \omega, \lambda(\nabla_a \lambda^{-1}(v)) \rangle, \end{aligned} \quad (137)$$

but, by recalling the fundamental property of the *duality adjoint* and by using once again Eq.(133), the second term in Eq.(137) can be written

$$\begin{aligned} \langle \omega, \lambda(\nabla_a \lambda^{-1}(v)) \rangle &= \langle \lambda^\Delta(\omega), \nabla_a \lambda^{-1}(v) \rangle \\ &= a \langle \lambda^\Delta(\omega), \lambda^{-1}(v) \rangle - \langle \nabla_a \lambda^\Delta(\omega), \lambda^{-1}(v) \rangle \\ &= a \langle \omega, v \rangle - \langle \lambda^{-\Delta} \nabla_a \lambda^\Delta(\omega), v \rangle, \end{aligned} \quad (138)$$

Finally, putting Eq.(138) into Eq.(137), the expected result follows. ■

4.4.1 Deformed Covariant Derivative of Multivector and Multiform Fields

Recall that λ^Δ is the so-called *duality adjoint* of λ , and $\lambda^{-\Delta}$ is a *short notation* for $(\lambda^\Delta)^{-1} = (\lambda^{-1})^\Delta$. We give now two properties of $\overset{\lambda}{\nabla}_a$ which are generalizations of the basic properties (135) and (136) above.

Proposition 33 For all $X \in \bigwedge \mathcal{V}(U)$

$$\overset{\lambda}{\nabla}_a X = \underline{\Delta}(\nabla_a \underline{\Delta}^{-1}(X)), \quad (139)$$

where $\underline{\Delta}$ is the so-called exterior power extension of λ , and $\underline{\Delta}^{-1}$ is a more simple notation for $(\underline{\Delta})^{-1} = \underline{(\lambda^{-1})}$.

For all $\Phi \in \bigwedge \mathcal{V}^*(U)$

$$\overset{\lambda}{\nabla}_a \Phi = \underline{\Delta}^{-\Delta}(\nabla_a \underline{\Delta}^\Delta(\Phi)), \quad (140)$$

where $\underline{\Delta}^\Delta = (\underline{\Delta})^\Delta = \underline{(\lambda^\Delta)}$ and $\underline{\Delta}^{-\Delta} = (\underline{\Delta}^\Delta)^{-1}$.

Proof. To prove the property for smooth multivector fields as given by Eq.(139), we make use the following criterion:

- (i) We check the statement for scalar fields $f \in \mathcal{S}(U)$ (use Eq.(99)).
- (ii) We check the statement for simple k -vector fields $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k \mathcal{V}(U)$ (use mathematical induction).
- (iii) We check the statement for a finite addition of simple k -vector fields $X^k + \cdots + Z^k \in \bigwedge^k \mathcal{V}(U)$ (use Eq.(104) and recalling the linear operator character for the extended of a linear operator)
- (iv) Then it becomes easy to prove the statement for multivector fields $X \in \bigwedge \mathcal{V}(U)$. ■

4.4.2 Deformed Covariant Derivative of Extensor Fields

We present now two properties for $\overset{\lambda}{\nabla}_a$ which are generalizations of the properties (139) and (140).

Proposition 34 For all $\tau \in \overset{l}{ext}_k \mathcal{V}(U)$:

$$\overset{\lambda}{\nabla}_a \tau = \underline{\Delta} \nabla_a \underline{\Delta}^{-1} \tau, \quad (141)$$

where $\underline{\Delta}^{-1} \tau$ means the action of $\underline{\Delta}^{-1}$ on the smooth multivector extensor field τ , and $\underline{\Delta} \nabla_a \underline{\Delta}^{-1} \tau$ is the action of $\underline{\Delta}$ on the smooth multivector extensor field $\nabla_a \underline{\Delta}^{-1} \tau$.

Proposition 35 For all $v \in \text{ext}_k^* \mathcal{V}(U)$:

$$\overset{\lambda}{\nabla}_a v = \underline{\Delta}^{-\Delta} \nabla_a \underline{\Delta}^\Delta v, \quad (142)$$

where $\underline{\Delta}^\Delta v$ means the action of $\underline{\Delta}^\Delta$ on the smooth multiform extensor field v , and $\underline{\Delta}^{-\Delta} \nabla_a \underline{\Delta}^\Delta v$ is the action of $\underline{\Delta}^{-\Delta}$ on the smooth multiform extensor field $\nabla_a \underline{\Delta}^\Delta v$.

Proof. We will prove only the property for smooth multivector extensor fields as given by Eq.(141). Without restrictions on generality, we can work with a multivector extensor field $(X, \Phi) \mapsto \tau(X, \Phi)$.

Then, using Eq.(118) and taking into account the properties (139) and (140), we can write

$$\begin{aligned} & (\overset{\lambda}{\nabla}_a \tau)(X, \Phi) \\ &= \overset{\lambda}{\nabla}_a \tau(X, \Phi) - \tau(\overset{\lambda}{\nabla}_a X, \Phi) - \tau(X, \overset{\lambda}{\nabla}_a \Phi) \\ &= \underline{\Delta}(\nabla_a \underline{\Delta}^{-1} \circ \tau(X, \Phi)) - \tau(\underline{\Delta}(\nabla_a \underline{\Delta}^{-1}(X)), \Phi) - \tau(X, \underline{\Delta}^{-\Delta}(\nabla_a \underline{\Delta}^\Delta(\Phi))), \end{aligned}$$

i.e.,

$$\begin{aligned} & (\underline{\Delta}^{-1} \circ \overset{\lambda}{\nabla}_a \tau)(X, \Phi) \\ &= \nabla_a \underline{\Delta}^{-1} \circ \tau(X, \Phi) - \underline{\Delta}^{-1} \circ \tau(\underline{\Delta}(\nabla_a \underline{\Delta}^{-1}(X)), \Phi) \\ & \quad - \underline{\Delta}^{-1} \circ \tau(X, \underline{\Delta}^{-\Delta}(\nabla_a \underline{\Delta}^\Delta(\Phi))). \end{aligned}$$

By recalling the action of an extended operator $\underline{\Delta}^{-1}$ on a multivector extensor τ , we get

$$\begin{aligned} & (\underline{\Delta}^{-1} \circ \overset{\lambda}{\nabla}_a \tau)(X, \Phi) \\ &= \nabla_a \underline{\Delta}^{-1} \tau(\underline{\Delta}^{-1}(X), \underline{\Delta}^\Delta(\Phi)) - \underline{\Delta}^{-1} \tau(\nabla_a \underline{\Delta}^{-1}(X), \underline{\Delta}^\Delta(\Phi)) \\ & \quad - \underline{\Delta}^{-1} \tau(\underline{\Delta}^{-1}(X), \nabla_a \underline{\Delta}^\Delta(\Phi)). \end{aligned}$$

Using once again Eq.(118), we have

$$(\overset{\lambda}{\nabla}_a \tau)(X, \Phi) = \underline{\Delta} \circ (\nabla_a \underline{\Delta}^{-1} \tau)(\underline{\Delta}^{-1}(X), \underline{\Delta}^\Delta(\Phi)),$$

and finally recalling once again the action of an exterior power extension operator $\underline{\Delta}$ on a multivector extensor $\nabla_a \underline{\Delta}^{-1} \tau$, the required result follows. ■

4.5 Relative Covariant Derivative

Definition 36 Let $\{b_\mu, \beta^\mu\}$ be a pair of dual frame fields for $U \subseteq M$. Associated with $\{b_\mu, \beta^\mu\}$ we can construct a well-defined connection on U given by the mapping

$$B : \mathcal{V}(U) \times \mathcal{V}(U) \longrightarrow \mathcal{V}(U),$$

such that

$$B(a, v) = [a\beta^\sigma(v)] b_\sigma. \quad (143)$$

B is called *relative connection on U with respect to $\{b_\mu, \beta^\mu\}$* (or simply *relative connection for short*).

The parallelism structure $\langle U, B \rangle$ will be called *relative parallelism structure* with respect to $\{b_\mu, \beta^\mu\}$.

The a -DCDO induced by the relative connection will be denoted by ∂_a . According with Eq.(132), the a -DCD of a smooth vector field is given by

$$\partial_a v = [a\beta^\sigma(v)] b_\sigma. \quad (144)$$

Note that ∂_a is the unique a -DCDO which satisfies the condition

$$\partial_a b_\mu = 0. \quad (145)$$

On the other hand, given a parallelism structure $\langle U_0, \Gamma \rangle$ and any relative parallelism structure $\langle U, B \rangle$, such that $U_0 \cap U \neq \emptyset$. There exists a smooth *2-covariant vector extensor field* on $U_0 \cap U$, namely γ , defined by

$$\mathcal{V}(U_0 \cap U) \times \mathcal{V}(U_0 \cap U) \ni (a, v) \mapsto \gamma(a, v) \in \mathcal{V}(U_0 \cap U)$$

such that

$$\gamma(a, v) = \beta^\mu(v) \nabla_a b_\mu \quad (146)$$

which satisfies

$$\Gamma(a, v) = B(a, v) + \gamma(a, v). \quad (147)$$

Such a extensor field γ will be called the *relative connection extensor field*³ on $U_0 \cap U$.

From Eq.(132), this means that for all $v \in \mathcal{V}(U_0 \cap U)$:

$$\nabla_a v = \partial_a v + \gamma_a(v), \quad (148)$$

where ∇_a is the a -DCDO associated with $\langle U_0, \Gamma \rangle$ and ∂_a is the a -DCDO associated with $\langle U, B \rangle$ (note that γ_a is a smooth *vector operator field* on $U_0 \cap U$ defined by $\gamma_a(v) = \gamma(a, v)$).

By using Eq.(133) and Eq.(148), we get that for all $\omega \in \mathcal{V}^*(U_0 \cap U)$:

$$\nabla_a \omega = \partial_a \omega - \gamma_a^\Delta(\omega), \quad (149)$$

where γ_a^Δ is the *dual adjoint* of γ_a (i.e., $\langle \gamma_a^\Delta(\omega), v \rangle = \langle \omega, \gamma_a(v) \rangle$).

³The properties of the tensor field $\gamma_{\mu\nu}^\alpha$ such that $\gamma(\partial_\mu, \partial_\nu) = \gamma_{\mu\nu}^\alpha \partial_\alpha$ where $\{\partial_\mu\}$ is a basis for $\mathcal{V}(U_0 \cap U)$ are studied in details in [27].

4.5.1 Relative Covariant Derivative of Multivector and Multiform Fields

Let $\langle U_0, \Gamma \rangle$ be a parallelism structure on U_0 , and let ∇_a be its associated a -DCDO. Take any relative parallelism structure $\langle U, B \rangle$ compatible with $\langle U_0, \Gamma \rangle$ (i.e., $U_0 \cap U \neq \emptyset$).

We present now the split theorem, i.e. a generalization of Eqs. (148) and (149), for smooth multivector fields and for multiform fields. We know that for all $v \in \mathcal{V}(U_0 \cap U)$, $\nabla_a v = \partial_a v + \gamma_a(v)$, and for all $\omega \in \mathcal{V}^*(U_0 \cap U)$, $\nabla_a \omega = \partial_a \omega - \gamma_a^\Delta(\omega)$.

Theorem 37 (a) For all $X \in \bigwedge \mathcal{V}(U_0 \cap U)$

$$\nabla_a X = \partial_a X + \gamma_a(X), \quad (150)$$

where γ_a is contracted extension operator⁴ of γ_a

(b) For all $\Phi \in \bigwedge \mathcal{V}^*(U_0 \cap U)$

$$\nabla_a \Phi = \partial_a \Phi - \gamma_a^\Delta(\Phi), \quad (151)$$

where γ_a^Δ is the so-called contracted extension operator of γ_a^Δ which as we know coincides with the so-called duality adjoint of γ_a .

Proof. To prove the property for smooth multivector fields as given by Eq.(150), we use as above the following procedure:.

- (i) We check the statement for scalar fields $f \in \mathcal{S}(U)$.
- (ii) Next, we check the statement for simple k -vector fields $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k \mathcal{V}(U)$.
- (iii) Next we check the statement for a finite addition of simple k -vector fields $X^k + \cdots Z^k \in \bigwedge^k \mathcal{V}(U)$.
- (iv) We then can easily prove the statement for multivector fields $X \in \bigwedge \mathcal{V}(U)$. ■

Let $\langle U, B \rangle$ and $\langle U', B' \rangle$, $U \cap U' \neq \emptyset$, be two compatible parallelism structures taken on a smooth manifold M . The a -DCDO's associated with $\langle U, B \rangle$ and $\langle U', B' \rangle$ are denoted by ∂_a and ∂'_a , respectively. As we already know [5], there exists a well-defined smooth extensor operator field on $U \cap U'$, the *Jacobian field* J (see Appendix E), such that the following two basic properties are satisfied: for all $v \in \mathcal{V}(U \cap U')$, $\partial'_a v = J(\partial_a J^{-1}(v))$, and for all $\omega \in \mathcal{V}^*(U \cap U')$, $\partial'_a \omega = J^{-\Delta}(\partial_a J^\Delta(\omega))$.

We can see immediately that the basic properties just recalled implies that ∂'_a is the J -deformation of ∂_a .

We present now two properties for the relative covariant derivatives which are generalizations of the basic properties just recalled above.

⁴See appendix 6.3 and [4], to recall the notion of the generalization procedure.

Proposition 38 For all $X \in \bigwedge \mathcal{V}(U \cap U')$

$$\partial'_a X = \underline{J}(\partial_a \underline{J}^{-1}(X)). \quad (152)$$

For all $\Phi \in \bigwedge \mathcal{V}^*(U \cap U')$

$$\partial'_a \Phi = \underline{J}^{-\Delta}(\partial_a \underline{J}^\Delta(\Phi)). \quad (153)$$

Proof. Eq.(152) is an immediate consequence of Eq.(139)., Eq.(153) is an immediate consequence of Eq.(140). ■

4.5.2 Relative Covariant Derivative of Extensor Fields

As we know from (150) and (151), there exists a well-defined smooth vector operator field on $U_0 \cap U$, called the *relative connection field* γ_a , which satisfies the *split theorem* valid for smooth multivector fields and for smooth multiform fields, i.e.: for all $X \in \bigwedge \mathcal{V}(U_0 \cap U)$, $\nabla_a X = \partial_a X + \gamma_a(X)$, and for all $\Phi \in \bigwedge \mathcal{V}^*(U_0 \cap U)$, $\nabla_a \Phi = \partial_a \Phi - \gamma_a^\Delta(\Phi)$.

We now present a split theorem for smooth multivector extensor fields and for smooth multiform extensor fields, which are the generalizations of the properties just recalled above.

Theorem 39 (i) For all $\tau \in \text{ext}_k^l \mathcal{V}(U_0 \cap U)$:

$$\nabla_a \tau = \partial_a \tau + \gamma_a \tau, \quad (154)$$

where $\gamma_a \tau$ means the action of γ_a on the smooth multivector extensor field τ .

(ii) For all $v \in \text{ext}_k^{*l} \mathcal{V}(U_0 \cap U)$:

$$\nabla_a v = \partial_a v - \gamma_a^\Delta v, \quad (155)$$

where $\gamma_a^\Delta v$ means the action of γ_a^Δ on the smooth multiform extensor field v .

Proof. We prove only the property for smooth multivector extensor fields, i.e., Eq.(154). Without loss of generality, we check this statement for a multivector extensor field $(X, \Phi) \mapsto \tau(X, \Phi)$.

Using Eq.(118), and by taking into account the properties just recalled above, we have

$$\begin{aligned} \nabla_a \tau(X, \Phi) &= \nabla_a \tau(X, \Phi) - \tau(\nabla_a X, \Phi) - \tau(X, \nabla_a \Phi) \\ &= \partial_a \tau(X, \Phi) + \gamma_a(\tau(X, \Phi)) \\ &\quad - \tau(\partial_a X + \gamma_a(X), \Phi) - \tau(X, \partial_a \Phi - \gamma_a^\Delta(\Phi)) \\ &= \partial_a \tau(X, \Phi) - \tau(\partial_a X, \Phi) - \tau(X, \partial_a \Phi) \\ &\quad + \gamma_a(\tau(X, \Phi)) - \tau(\gamma_a(X), \Phi) + \tau(X, \gamma_a^\Delta(\Phi)). \end{aligned}$$

Then, using once again Eq.(118) and recalling the action of a generalized operator γ_a on a multivector extensor τ , see Eq.(61), we get

$$(\nabla_a \tau)(X, \Phi) = (\partial_a \tau)(X, \Phi) + (\gamma_a \tau)(X, \Phi),$$

and the proposition is proved. ■

5 Torsion and Curvature

Definition 40 Let $\langle U, \Gamma \rangle$ be a parallelism structure on U . The smooth 2-covariant vector extensor field on U , defined by

$$\mathcal{V}(U) \times \mathcal{V}(U) \ni (a, b) \mapsto \tau(a, b) \in \mathcal{V}(U)$$

such that

$$\tau(a, b) = \nabla_a b - \nabla_b a - [a, b], \quad (156)$$

will be called the fundamental torsion extensor field⁵ of $\langle U, \Gamma \rangle$.

Accordingly, there exists a smooth $(2, 1)$ -extensor field on U , defined by

$$\bigwedge^2 \mathcal{V}(U) \ni X^2 \mapsto \mathcal{T}(X^2) \in \mathcal{V}(U)$$

such that

$$\mathcal{T}(X^2) = \frac{1}{2} \langle \varepsilon^\mu \wedge \varepsilon^\nu, X^2 \rangle \tau(e_\mu, e_\nu), \quad (157)$$

where $\{e_\mu, \varepsilon^\mu\}$ is any pair of dual frame fields on $V \supseteq U$. It should be emphasized that \mathcal{T} , as extensor field associated with τ , is well-defined since the vector field $\mathcal{T}(X^2)$ does not depend on the choice of $\{e_\mu, \varepsilon^\mu\}$.

It should be remarked that the well known *torsion tensor field*, see, e.g., [1, 27], is just given by

$$\mathcal{V}(U) \times \mathcal{V}(U) \times \mathcal{V}^*(U) \ni (a, b, \omega) \mapsto T(a, b, \omega) \in \mathcal{S}(U)$$

such that

$$T(a, b, \omega) = \langle \omega, \tau(a, b) \rangle. \quad (158)$$

Note that it is possible to get τ in terms of T , i.e.,

$$\tau(a, b) = T(a, b, \varepsilon^\mu) e_\mu. \quad (159)$$

It is also possible to introduce a *third torsion extensor field* for $\langle U, \Gamma \rangle$ by defining the smooth $(1, 2)$ -extensor field on U ,

$$\mathcal{V}^*(U) \ni \omega \mapsto \Theta(\omega) \in \bigwedge^2 \mathcal{V}^*(U),$$

⁵ τ is skew-symmetric, i.e.,

$$\tau(b, a) = -\tau(a, b).$$

such that

$$\Theta(\omega) = \frac{1}{2} \langle \omega, \tau(e_\mu, e_\nu) \rangle \varepsilon^\mu \wedge \varepsilon^\nu, \quad (160)$$

We call Θ the *Cartan torsion extensor field* of $\langle U, \Gamma \rangle$.

Definition 41 *The smooth 3-covariant extensor vector field on U , defined by*

$$\mathcal{V}(U) \times \mathcal{V}(U) \times \mathcal{V}(U) \ni (a, b, c) \mapsto \rho(a, b, c) \in \mathcal{V}(U),$$

such that

$$\rho(a, b, c) = [\nabla_a, \nabla_b]c - \nabla_{[a, b]}c, \quad (161)$$

will be called the fundamental curvature extensor field⁶ of $\langle U, \Gamma \rangle$.

Thus, there exists a smooth 1 *bivector and 1 vector variables vector extensor field* on U , defined by

$$\bigwedge^2 \mathcal{V}(U) \times \mathcal{V}(U) \ni (X^2, c) \mapsto \mathcal{R}(X^2, c) \in \mathcal{V}(U)$$

such that

$$\mathcal{R}(X^2, c) = \frac{1}{2} \langle \varepsilon^\mu \wedge \varepsilon^\nu, X^2 \rangle \rho(e_\mu, e_\nu, c), \quad (162)$$

where $\{e_\mu, \varepsilon^\mu\}$ is any pair of dual frame fields on $V \supseteq U$. We note that \mathcal{R} , as extensor field associated with ρ , is well-defined since the vector field $\mathcal{R}(X^2, c)$ does not depend on the choice of $\{e_\mu, \varepsilon^\mu\}$.

Proposition 42 *The fundamental curvature field ρ satisfies a cyclic property, i.e.,*

$$\rho(a, b, c) + \rho(b, c, a) + \rho(c, a, b) = 0.$$

Proof. Let us take $a, b, c \in \mathcal{V}(U)$. By using Eq.(161), we can write

$$\rho(a, b, c) = \nabla_a \nabla_b c - \nabla_b \nabla_a c - \nabla_{[a, b]}c, \quad (a)$$

$$\rho(b, c, a) = \nabla_b \nabla_c a - \nabla_c \nabla_b a - \nabla_{[b, c]}a, \quad (b)$$

$$\rho(c, a, b) = \nabla_c \nabla_a b - \nabla_a \nabla_c b - \nabla_{[c, a]}b. \quad (c)$$

By adding Eqs. (a), (b) and (c), we have

$$\begin{aligned} & \rho(a, b, c) + \rho(b, c, a) + \rho(c, a, b) \\ &= \nabla_a(\nabla_b c - \nabla_c b) + \nabla_b(\nabla_c a - \nabla_a c) + \nabla_c(\nabla_a b - \nabla_b a) \\ & \quad - \nabla_{[a, b]}c - \nabla_{[b, c]}a - \nabla_{[c, a]}b, \end{aligned} \quad (d)$$

but, by taking into account Eq.(164), we get

$$\rho(a, b, c) + \rho(b, c, a) + \rho(c, a, b) = [a, [b, c]] + [b, [c, a]] + [c, [a, b]], \quad (e)$$

⁶As can be easily verified, ρ is skew-symmetric with respect to the first and the second variables, i.e., $\rho(b, a, c) = -\rho(a, b, c)$.

whence, by recalling the Jacobi identities for the Lie product of smooth vector fields, the expected result immediately follows. ■

The fundamental curvature field ρ satisfies the so-called *Bianchi identity*, i.e.,

$$\nabla_w \rho(a, b, c) + \nabla_a \rho(b, w, c) + \nabla_b \rho(w, a, c) = 0.$$

Note the cycling of letters: $a, b, w \rightarrow b, w, a \rightarrow w, a, b$. The proof is a single exercise, see, e.g., [5].

5.1 Symmetric Parallelism Structure

A parallelism structure $\langle U, \Gamma \rangle$ is said to be *symmetric* if and only if for all smooth vector fields a and b on U it holds

$$\Gamma(a, b) - \Gamma(b, a) = [a, b], \quad (163)$$

i.e.,

$$\nabla_a b - \nabla_b a = [a, b]. \quad (164)$$

Now, according with Eq.(156), we see that the *condition of symmetry* is completely equivalent to the *condition of null torsion*, i.e.,

$$\tau(a, b) = 0. \quad (165)$$

So, taking into account Eq.(157) and Eq.(160), we also have that

$$\mathcal{T}(X^2) = 0 \text{ and } \Theta(\omega) = 0. \quad (166)$$

6 Appendix

A Multivectors and Multiforms

Let V be a vector space over \mathbb{R} with finite dimension, i.e., $\dim V = n$ with $n \in \mathbb{N}$, and let V^* be its dual vector space. Recall that

$$\dim V = \dim V^* = n. \quad (167)$$

Let us consider an integer number k with $0 \leq k \leq n$. The real vector spaces of k -vectors over V , i.e., the set of skew-symmetric k -contravariant tensors over V , and the real vector spaces of k -forms over V , i.e., the set of skew-symmetric k -covariant tensors over V , will be as usually denoted by $\bigwedge^k V$ and $\bigwedge^k V^*$, respectively.

We identify, as usual 0-vectors to real numbers, i.e., $\bigwedge^0 V = \mathbb{R}$, and 1-vectors to objects living in V , i.e., $\bigwedge^1 V \simeq V$. Also, we identify 0-forms with real numbers, i.e., $\bigwedge^0 V^* = \mathbb{R}$, and 1-forms with objects living in V^* , i.e., $\bigwedge^1 V^* = V^*$. Recall that

$$\dim \bigwedge^k V = \dim \bigwedge^k V^* = \binom{n}{k}. \quad (168)$$

The 0-vectors, 2-vectors, ..., $(n-1)$ -vectors and n -vectors are called scalars, bivectors, ..., pseudovectors and pseudoscalars, respectively. The 0-forms, 2-forms, ..., $(n-1)$ -forms and n -forms are called scalars, biforms, ..., pseudoforms and pseudoscalars.

Given a vector space V over the real field \mathbb{R} , we define $\bigwedge V$ as the exterior direct sum

$$\bigwedge V = \sum_{r=0}^n \oplus \bigwedge^r V = \bigoplus_{r=0}^n \bigwedge^r V.$$

To simplify the notation we sometimes write *simply*

$$\bigwedge V = \mathbb{R} + V + \bigwedge^2 V + \cdots + \bigwedge^{n-1} V + \bigwedge^n V.$$

As it is well known the set of multivectors over V has a natural structure of vector space over \mathbb{R} and we have

$$\begin{aligned} \dim \bigwedge V &= \dim \mathbb{R} + \dim V + \dim \bigwedge^2 V + \cdots + \dim \bigwedge^{n-1} V + \dim \bigwedge^n V \\ &= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n. \end{aligned} \quad (169)$$

An element of $\bigwedge V$ will be called a multivector over V . If $X \in \bigwedge V$ we write:

$$X = X^0 + X^1 + X^2 + \cdots + X^{n-1} + X^n.$$

In what follows we shall need also the vector space $\bigwedge V^* = \bigoplus_{r=0}^n \bigwedge^r V^*$. An element of $\bigwedge V^*$ will be called a *multiform over V* . As in the case of multivectors we simply write:

$$\bigwedge V^* = \mathbb{R} + V^* + \bigwedge^2 V^* + \cdots + \bigwedge^{n-1} V^* + \bigwedge^n V^*,$$

and if $\Phi \in \bigwedge V^*$ we write

$$\Phi = \Phi_0 + \Phi_1 + \Phi_2 + \cdots + \Phi_{n-1} + \Phi_n.$$

Of course, $\bigwedge V^*$ has a natural structure of real vector space over \mathbb{R} . We have,

$$\begin{aligned} \dim \bigwedge V^* &= \dim \mathbb{R} + \dim V^* + \dim \bigwedge^2 V^* + \cdots + \dim \bigwedge^{n-1} V^* + \dim \bigwedge^n V^* \\ &= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n. \end{aligned} \quad (170)$$

We recall that $\bigwedge^k V$ is also called the homogeneous multivector space (of degree k), and to $\bigwedge^k V^*$ the homogeneous multiform space (of degree k).

Let us take an integer number k with $0 \leq k \leq n$. The linear mappings

$$\bigwedge V \ni X \mapsto \langle X \rangle^k \in \bigwedge V \text{ and } \bigwedge V^* \ni \Phi \mapsto \langle \Phi \rangle_k \in \bigwedge V^*$$

such that if $X = X^0 + X^1 + \cdots + X^n$ and $\Phi = \Phi_0 + \Phi_1 + \cdots + \Phi_n$, then

$$\langle X \rangle^k = X^k \text{ and } \langle \Phi \rangle_k = \Phi_k \quad (171)$$

are called the *k-part operator (for multivectors)* and the *k-part operator (for multiforms)*, respectively. $\langle X \rangle^k$ is read as the *k-part of X* and $\langle \Phi \rangle_k$ is read as the *k-part of Φ*.

It should be evident that for all $X \in \bigwedge V$ and $\Phi \in \bigwedge V^*$:

$$X = \sum_{k=0}^n \langle X \rangle^k, \quad (172)$$

$$\Phi = \sum_{k=0}^n \langle \Phi \rangle_k. \quad (173)$$

The linear mappings

$$\bigwedge V \ni X \mapsto \widehat{X} \in \bigwedge V \text{ and } \bigwedge V^* \ni \Phi \mapsto \widehat{\Phi} \in \bigwedge V^*$$

such that

$$\langle \widehat{X} \rangle^k = (-1)^k \langle X \rangle^k \text{ and } \langle \widehat{\Phi} \rangle_k = (-1)^k \langle \Phi \rangle_k \quad (174)$$

are called the *grade involution operator (for multivectors)* and the *grade involution operator (for multiforms)*, respectively.

The linear mappings

$$\bigwedge V \ni X \mapsto \widetilde{X} \in \bigwedge V \text{ and } \bigwedge V^* \ni \Phi \mapsto \widetilde{\Phi} \in \bigwedge V^*$$

such that

$$\langle \widetilde{X} \rangle^k = (-1)^{\frac{1}{2}k(k-1)} \langle X \rangle^k \text{ and } \langle \widetilde{\Phi} \rangle_k = (-1)^{\frac{1}{2}k(k-1)} \langle \Phi \rangle_k \quad (175)$$

are called the *reversion operator (for multivectors)* and the *reversion operator (for multiforms)*, respectively.

Both of $\bigwedge V$ and $\bigwedge V^*$ endowed with the exterior product \wedge (of multivectors and multiforms!) are *associative algebras*, i.e., the *exterior algebra of multivectors* and the *exterior algebra of multiforms*, respectively.

B Exterior Power Extension for Operators

Let λ be a linear operator on V , i.e., a linear map $V \ni v \mapsto \lambda(v) \in V$. It can be extended in such a way as to give a linear operator on $\bigwedge V$, namely $\underline{\lambda}$, which is defined by

$$\bigwedge V \ni X \mapsto \underline{\lambda}(X) \in \bigwedge V$$

such that

$$\underline{\lambda}(X) = \langle 1, X \rangle + \sum_{k=1}^n \frac{1}{k!} \langle \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_k}, X \rangle \lambda(e_{j_1}) \wedge \cdots \wedge \lambda(e_{j_k}), \quad (176)$$

where $\{e_j, \varepsilon^j\}$ is any pair of dual bases for V and V^* .

We emphasize that $\underline{\lambda}$ is a well-defined linear operator on $\bigwedge V$. Note that each k -vector $\langle \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_k}, X \rangle \lambda(e_{j_1}) \wedge \cdots \wedge \lambda(e_{j_k})$ with $1 \leq k \leq n$ does not depend

on the choice of $\{e_j, \varepsilon^j\}$, and the linearity follows just from the linearity of the duality scalar product. We call $\underline{\lambda}$ the *Exterior Power Extension (EPE)* of λ (to multivector operator).

The EPE of a vector operator λ has the following basic properties:

$\underline{\lambda}$ is grade-preserving, i.e.,

$$\text{if } X \in \bigwedge^k V, \text{ then } \underline{\lambda}(X) \in \bigwedge^k V. \quad (177)$$

For all $\alpha \in \mathbb{R}$, $v \in V$, and $X, Y \in \bigwedge V$:

$$\underline{\lambda}(\alpha) = \alpha, \quad (178)$$

$$\underline{\lambda}(v) = \lambda(v), \quad (179)$$

$$\underline{\lambda}(X \wedge Y) = \underline{\lambda}(X) \wedge \underline{\lambda}(Y). \quad (180)$$

We observe that the four basic properties as given by Eq.(177), Eq.(178), Eq.(179) and Eq.(180) are completely equivalent to the extension procedure of a vector operator.

Let λ be a linear operator on V^* , i.e., a linear map $V^* \ni \omega \mapsto \lambda(\omega) \in V^*$. It is possible to extend λ in such a way to get a linear operator on $\bigwedge V^*$, namely the operator $\underline{\lambda}$, defined by

$$\bigwedge V^* \ni \Phi \mapsto \underline{\lambda}(\Phi) \in \bigwedge V^*,$$

such that

$$\underline{\lambda}(\Phi) = \langle 1, \Phi \rangle + \sum_{k=1}^n \frac{1}{k!} \langle e_{j_1} \wedge \cdots \wedge e_{j_k}, \Phi \rangle \lambda(\varepsilon^{j_1}) \wedge \cdots \wedge \lambda(\varepsilon^{j_k}), \quad (181)$$

where $\{e_j, \varepsilon^j\}$ is any pair of dual bases for V and V^* .

We emphasize that $\underline{\lambda}$ is a well-defined linear operator on $\bigwedge V^*$. We call $\underline{\lambda}$ the *EPE* of λ (to multiforms).

The EPE of a form operator λ has the following basic properties.

$\underline{\lambda}$ is grade-preserving, i.e.,

$$\text{if } \Phi \in \bigwedge^k V^*, \text{ then } \underline{\lambda}(\Phi) \in \bigwedge^k V^*. \quad (182)$$

For all $\alpha \in \mathbb{R}$, $\omega \in V^*$, and $\Phi, \Psi \in \bigwedge V^*$:

$$\underline{\lambda}(\alpha) = \alpha, \quad (183)$$

$$\underline{\lambda}(\omega) = \lambda(\omega), \quad (184)$$

$$\underline{\lambda}(\Phi \wedge \Psi) = \underline{\lambda}(\Phi) \wedge \underline{\lambda}(\Psi). \quad (185)$$

The four basic properties given by Eq.(182), Eq.(183) Eq.(184) and Eq.(185) are logically equivalent to the extension procedure of a form operator.

There exists a relationship between the extension procedure of a vector operator and the extension procedure of a form operator.

Let us take a vector operator (or, a form operator) λ . As we can see, the duality adjoint of λ is just a form operator (respectively, a vector operator), and

the duality adjoint of $\underline{\lambda}$ is just a multiform operator (respectively, a multivector operator). It holds that the duality adjoint of the EPE of λ is equal to the EPE of the duality adjoint of λ , i.e.,

$$(\underline{\lambda})^\Delta = \underline{(\lambda^\Delta)}. \quad (186)$$

Thus, it is possible to use the more simple notation $\underline{\lambda}^\Delta$ to mean either $(\underline{\lambda})^\Delta$ or $\underline{(\lambda^\Delta)}$.

We present some properties for the EPE of an invertible vector operator λ . For all $\Phi \in \bigwedge V^*$, and $X \in \bigwedge V$:

$$\underline{\lambda} \langle \Phi, X \rangle = \langle \underline{\lambda}^{-\Delta}(\Phi), \underline{\lambda}(X) \rangle, \quad (187)$$

$$\underline{\lambda} \langle \Phi, X | = \langle \underline{\lambda}^{-\Delta}(\Phi), \underline{\lambda}(X) |, \quad (188)$$

$$\underline{\lambda} |X, \Phi \rangle = | \underline{\lambda}(X), \underline{\lambda}^{-\Delta}(\Phi) \rangle. \quad (189)$$

We present only the proof for the property given by Eq.(188), the other proofs are analogous.

Proof. Let us take $X \in \bigwedge V$ and $\Phi, \Psi \in \bigwedge V^*$. A straightforward calculation, using Eq.(38), Eq.(24), Eq.(185) and Eq.(182), yields

$$\begin{aligned} \langle \underline{\lambda} \langle \Phi, X |, \Psi \rangle &= \langle \langle \Phi, X |, \underline{\lambda}^\Delta(\Psi) \rangle = \langle X, \tilde{\Phi} \wedge \underline{\lambda}^\Delta(\Psi) \rangle \\ &= \langle X, \underline{\lambda}^\Delta(\underline{\lambda}^{-\Delta}(\tilde{\Phi}) \wedge \Psi) \rangle = \langle \underline{\lambda}(X), \widetilde{\underline{\lambda}^{-\Delta}(\Phi)} \wedge \Psi \rangle \\ &= \langle \langle \underline{\lambda}^{-\Delta}(\Phi), \underline{\lambda}(X) |, \Psi \rangle, \end{aligned}$$

whence, by the non-degeneracy of duality scalar product the result follows. ■

We present now some properties for the EPE of an invertible form operator λ .

For all $\Phi \in \bigwedge V^*$, and $X \in \bigwedge V$:

$$\underline{\lambda} \langle \Phi, X \rangle = \langle \underline{\lambda}(\Phi), \underline{\lambda}^{-\Delta}(X) \rangle, \quad (190)$$

$$\underline{\lambda} \langle \Phi, X | = \langle \underline{\lambda}(\Phi), \underline{\lambda}^{-\Delta}(X) |, \quad (191)$$

$$\underline{\lambda} |X, \Phi \rangle = | \underline{\lambda}^{-\Delta}(X), \underline{\lambda}(\Phi) \rangle. \quad (192)$$

C Contracted Extension for Operator

Let γ be a linear operator on V , i.e., a linear map $V \ni v \mapsto \gamma(v) \in V$. It can be generalized in such a way to give a linear operator on $\bigwedge V$, namely $\underline{\gamma}$, which is defined by

$$\bigwedge V \ni X \mapsto \underline{\gamma}(X) \in \bigwedge V$$

such that

$$\gamma(X) = \gamma(e_j) \wedge \langle \varepsilon^j, X |, \quad (193)$$

where $\{e_j, \varepsilon^j\}$ is any pair of dual bases for V and V^* .

We note that the multivector $\gamma(e_j) \wedge \langle \varepsilon^j, X |$ does not depend on the choice of $\{e_j, \varepsilon^j\}$, and that the linearity of the duality contracted product implies the linearity of γ . Thus, γ is a well-defined linear operator on $\bigwedge V$. We call γ the

Contracted Extension (CE) of γ (to multivector operator).

The CE of a vector operator γ has the following basic properties.

γ is grade-preserving, i.e.,

$$\text{if } X \in \bigwedge^k V, \text{ then } \gamma(X) \in \bigwedge^k V. \quad (194)$$

For all $\alpha \in \mathbb{R}$, $v \in V$, and $X, Y \in \bigwedge V$:

$$\gamma(\alpha) = 0, \quad (195)$$

$$\gamma(v) = \gamma(v), \quad (196)$$

$$\gamma(X \wedge Y) = \gamma(X) \wedge Y + X \wedge \gamma(Y). \quad (197)$$

The four properties given by Eq.(194), Eq.(195), Eq.(196) and Eq.(197) are completely equivalent to the generalization procedure for vector operators.

Let γ be a linear operator on V^* , i.e., a linear map $V^* \ni \omega \mapsto \gamma(\omega) \in V^*$. It is possible to generalize γ in such a way as to get a linear operator on $\bigwedge V^*$, namely γ , which is defined by

$$\bigwedge V^* \ni \Phi \mapsto \gamma(\Phi) \in \bigwedge V^*$$

such that

$$\gamma(\Phi) = \gamma(\varepsilon^j) \wedge \langle \varepsilon_j, \Phi |, \quad (198)$$

where $\{e_j, \varepsilon^j\}$ is any pair of dual bases over V .

We emphasize that γ is a well-defined linear operator on $\bigwedge V^*$, and call it the *CE of γ* (to a multiform operator).

The generalized of a form operator γ has the following basic properties.

γ is grade-preserving, i.e.,

$$\text{if } \Phi \in \bigwedge^k V^*, \text{ then } \gamma(\Phi) \in \bigwedge^k V^*. \quad (199)$$

For all $\alpha \in \mathbb{R}$, $\omega \in V^*$, and $\Phi, \Psi \in \bigwedge V^*$ we have

$$\gamma(\alpha) = 0, \quad (200)$$

$$\gamma(\omega) = \gamma(\omega), \quad (201)$$

$$\gamma(\Phi \wedge \Psi) = \gamma(\Phi) \wedge \Psi + \Phi \wedge \gamma(\Psi). \quad (202)$$

The properties given by Eq.(199), Eq.(200), Eq.(201) and Eq.(202) are logically equivalent to the generalization procedure for *form* operators.

There exists a relationship between the generalization procedure of a vector operator and the generalization procedure of a form operator.

Let γ a vector operator (or, a form operator). As we already know, the duality adjoint of γ is just a form operator (respectively, a vector operator), and the duality adjoint of γ is just a multiform operator (respectively, a multivector operator). The duality adjoint of the CE of γ is equal to the generalized of the duality adjoint of γ , i.e.,

$$\left(\gamma\right)^{\Delta} = (\gamma^{\Delta}). \quad (203)$$

It follows that is possible to use a more simple notation, namely γ^{Δ} to mean either $\left(\gamma\right)^{\Delta}$ or (γ^{Δ}) .

We give some of the main properties of the CE of a vector operator γ .

For all $\Phi \in \wedge V^*$, and $X \in \wedge V$:

$$\gamma \langle \Phi, X \rangle = - \left\langle \gamma^{\Delta}(\Phi), X \right\rangle + \left\langle \Phi, \gamma(X) \right\rangle, \quad (204)$$

$$\gamma \langle \Phi, X | = - \left\langle \gamma^{\Delta}(\Phi), X \right| + \left\langle \Phi, \gamma(X) \right|, \quad (205)$$

$$\gamma |X, \Phi \rangle = \left| \gamma(X), \Phi \right\rangle - \left| X, \gamma^{\Delta}(\Phi) \right\rangle. \quad (206)$$

We prove the property given by Eq.(205), the other proofs are analogous.

Proof. Let us take $X \in \wedge V$ and $\Phi, \Psi \in \wedge V^*$. A straightforward calculation, by using Eq.(38), Eq.(24), Eq.(199) and Eq.(202), yields

$$\begin{aligned} \left\langle \gamma \langle \Phi, X |, \Psi \right\rangle &= \left\langle \langle \Phi, X |, \gamma^{\Delta}(\Psi) \right\rangle = \left\langle X, \tilde{\Phi} \wedge \gamma^{\Delta}(\Psi) \right\rangle \\ &= \left\langle X, -\widetilde{\gamma^{\Delta}(\Phi)} \wedge \Psi + \gamma^{\Delta}(\tilde{\Phi}) \wedge \Psi + \tilde{\Phi} \wedge \gamma^{\Delta}(\Psi) \right\rangle \\ &= - \left\langle \left\langle \gamma^{\Delta}(\Phi), X \right|, \Psi \right\rangle + \left\langle X, \gamma^{\Delta}(\tilde{\Phi} \wedge \Psi) \right\rangle \\ &= - \left\langle \left\langle \gamma^{\Delta}(\Phi), X \right|, \Psi \right\rangle + \left\langle \left\langle \Phi, \gamma(X) \right|, \Psi \right\rangle \right\rangle \\ &= \left\langle - \left\langle \gamma^{\Delta}(\Phi), X \right| + \left\langle \Phi, \gamma(X) \right|, \Psi \right\rangle, \end{aligned}$$

and by the non-degeneracy of duality scalar product, the expected result follows. ■

We present some properties for the generalized of a form operator γ .

For all $\Phi \in \bigwedge V^*$, and $X \in \bigwedge V$:

$$\gamma \langle \Phi, X \rangle = \left\langle \gamma(\Phi), X \right\rangle - \left\langle \Phi, \gamma^\Delta(X) \right\rangle, \quad (207)$$

$$\gamma \langle \Phi, X | = \left\langle \gamma(\Phi), X \right| - \left\langle \Phi, \gamma^\Delta(X) \right|, \quad (208)$$

$$\gamma |X, \Phi \rangle = - \left| \gamma^\Delta(X), \Phi \right\rangle + \left| X, \gamma(\Phi) \right\rangle. \quad (209)$$

D Extensors

Extensors are a new kind of geometrical objects which play a crucial role in the theory presented here and in what follows the basics of their theory is described. These objects have been apparently introduced by Hestenes and Sobczyk in [16] and some applications of the concept appears in [18], but a rigorous theory was developed later, more details on the theory of extensors may be found in [11].

Let $\bigwedge_1^\diamond V, \dots$ and $\bigwedge_k^\diamond V$ be k subspaces of $\bigwedge V$ such that each of them is any sum of homogeneous subspaces of $\bigwedge V$, and let $\bigwedge_1^\diamond V^*, \dots$ and $\bigwedge_l^\diamond V^*$ be l subspaces of $\bigwedge V^*$ such that each of them is any sum of homogeneous subspaces of $\bigwedge V^*$.

If $\bigwedge^\diamond V$ is any sum of homogeneous subspaces of $\bigwedge V$, a multilinear mapping

$$\underbrace{\bigwedge_1^\diamond V \times \dots \times \bigwedge_k^\diamond V}_{k\text{-copies}} \times \underbrace{\bigwedge_1^\diamond V^* \times \dots \times \bigwedge_l^\diamond V^*}_{l\text{-copies}} \ni (X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) \\ \mapsto \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) \in \bigwedge^\diamond V \quad (210)$$

is called a k *multivector* and l *multiform variables multivector extensor* over V .

If $\bigwedge^\diamond V^*$ is any sum of homogeneous subspaces of $\bigwedge V^*$, a multilinear mapping

$$\underbrace{\bigwedge_1^\diamond V \times \dots \times \bigwedge_k^\diamond V}_{k\text{-copies}} \times \underbrace{\bigwedge_1^\diamond V^* \times \dots \times \bigwedge_l^\diamond V^*}_{l\text{-copies}} \ni (X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) \\ \mapsto \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) \in \bigwedge^\diamond V^* \quad (211)$$

is called a k *multivector* and l *multiform variables multiform extensor* over V .

The set of all the k multivector and l multiform variables multivector extensors over V has a natural structure of real vector space, and will be denoted by the highly suggestive notation $ext(\bigwedge_1^\diamond V, \dots, \bigwedge_k^\diamond V, \bigwedge_1^\diamond V^*, \dots, \bigwedge_l^\diamond V^*; \bigwedge^\diamond V)$.

When no confusion arises, we use the more simple notation $ext_k^l(V)$ for that space.

We obviously have that:

$$\dim ext_k^l(V) = \dim \bigwedge_1^\diamond V \dots \dim \bigwedge_k^\diamond V \dim \bigwedge_1^\diamond V^* \dots \dim \bigwedge_l^\diamond V^* \dim \bigwedge^\diamond V. \quad (212)$$

The set of all the k multivector and l multiform variables multiform extensors over V has also a natural structure of real vector space, and will be denoted by

$ext(\bigwedge_1^\diamond V, \dots, \bigwedge_k^\diamond V, \bigwedge_1^\diamond V^*, \dots, \bigwedge_l^\diamond V^*; \bigwedge^\diamond V^*)$, and when no confusion arises, we use the simple notation $ext_k^{*l}(V)$ for this space. Also, we have,

$$\dim ext_k^{*l}(V) = \dim \bigwedge_1^\diamond V \dots \dim \bigwedge_k^\diamond V \dim \bigwedge_1^\diamond V^* \dots \dim \bigwedge_l^\diamond V^* \dim \bigwedge^\diamond V^*. \quad (213)$$

In particular, when $\bigwedge_1^\diamond V = \dots = \bigwedge_k^\diamond V = V$ and $\bigwedge_1^\diamond V^* = \dots = \bigwedge_l^\diamond V^* = V^*$, we have two types of elementary extensors over a real vector space V , namely when $\bigwedge^\diamond V = V$ and when $\bigwedge^\diamond V^* = V^*$.

D.1 Exterior Product of Extensors

We define the exterior product of $\tau \in ext_k^l(V)$ and $\sigma \in ext_r^s(V)$ (or, $\tau \in ext_k^{*l}(V)$ and $\sigma \in ext_r^{*s}(V)$) as $\tau \wedge \sigma \in ext_{k+r}^{l+s}(V)$ (respectively, $\tau \wedge \sigma \in ext_{k+r}^{*l+s}(V)$) given by

$$\begin{aligned} \tau \wedge \sigma(X_1, \dots, X_k, Y_1, \dots, Y_r, \Phi^1, \dots, \Phi^l, \Psi^1, \dots, \Psi^s) \\ = \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) \wedge \sigma(Y_1, \dots, Y_r, \Psi^1, \dots, \Psi^s). \end{aligned} \quad (214)$$

Note that on the right side appears an exterior product of multivectors (respectively, an exterior product of multiforms).

The duality scalar product of a multiform extensor $\tau \in ext_k^{*l}(V)$ with a multivector extensor $\sigma \in ext_r^s(V)$ is the *scalar* extensor $\langle \tau, \sigma \rangle \in ext_{k+r}^{*l+s}(V)$ defined by

$$\begin{aligned} \langle \tau, \sigma \rangle(X_1, \dots, X_k, Y_1, \dots, Y_r, \Phi^1, \dots, \Phi^l, \Psi^1, \dots, \Psi^s) \\ = \langle \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l), \sigma(Y_1, \dots, Y_r, \Psi^1, \dots, \Psi^s) \rangle. \end{aligned} \quad (215)$$

The duality left contracted product of a multiform extensor $\tau \in ext_k^{*l}(V)$ with a multivector extensor $\sigma \in ext_r^s(V)$ (or, a multivector extensor $\tau \in ext_k^l(V)$ with a multiform extensor $\sigma \in ext_r^{*s}(V)$) is the multivector extensor $\langle \tau, \sigma | \in ext_{k+r}^{l+s}(V)$ (respectively, the multiform extensor $\langle \tau, \sigma | \in ext_{k+r}^{*l+s}(V)$) defined by

$$\begin{aligned} \langle \tau, \sigma | (X_1, \dots, X_k, Y_1, \dots, Y_r, \Phi^1, \dots, \Phi^l, \Psi^1, \dots, \Psi^s) \\ = \langle \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l), \sigma(Y_1, \dots, Y_r, \Psi^1, \dots, \Psi^s) \rangle|. \end{aligned} \quad (216)$$

The duality right contracted product of a multiform extensor $\tau \in ext_k^{*l}(V)$ with a multivector extensor $\sigma \in ext_r^s(V)$ (or, a multivector extensor $\tau \in ext_k^l(V)$ with a multiform extensor $\sigma \in ext_r^{*s}(V)$) is the multiform extensor

$|\tau, \sigma\rangle \in \text{ext}_{k+r}^{* \ l+s}(V)$ (respectively, the multivector extensor $|\tau, \sigma\rangle \in \text{ext}_{k+r}^{l+s}(V)$) defined by

$$\begin{aligned} & |\tau, \sigma\rangle (X_1, \dots, X_k, Y_1, \dots, Y_r, \Phi^1, \dots, \Phi^l, \Psi^1, \dots, \Psi^s) \\ &= |\tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l), \sigma(Y_1, \dots, Y_r, \Psi^1, \dots, \Psi^s)\rangle. \end{aligned} \quad (217)$$

E Jacobian Fields

Let $\{b_\mu, \beta^\mu\}$ and $\{b'_\mu, \beta^{\mu'}\}$ be any two pairs of dual frame fields on the open sets $U \subseteq M$ and $U' \subseteq M$, respectively.

If the parallelism structure $\langle U, B \rangle$ is compatible with the parallelism structure $\langle U', B' \rangle$ (i.e., define the same connection on $U \cap U' \neq \emptyset$), then we can define a smooth *extensor operator field* on $U \cap U'$, namely J , by

$$\mathcal{V}(U \cap U') \ni v \mapsto J(v) \in \mathcal{V}(U \cap U'),$$

such that

$$J(v) = \beta^\sigma(v) b'_\sigma. \quad (218)$$

It will be called the *Jacobian field* associated with the pairs of frame fields $\{b_\mu, \beta^\mu\}$ and $\{b'_\mu, \beta^{\mu'}\}$ (in this order!).

Note that in accordance with the above definition the Jacobian field associated with $\{b'_\mu, \beta^{\mu'}\}$ and $\{b_\mu, \beta^\mu\}$ is J' , given by

$$\mathcal{V}(U \cap U') \ni v \mapsto J'(v) \in \mathcal{V}(U \cap U'),$$

such that

$$J'(v) = \beta^{\sigma'}(v) b_\sigma. \quad (219)$$

It is the *inverse extensor operator* of J , i.e., $J \circ J'(v) = v$ and $J' \circ J(v) = v$ for each $v \in \mathcal{V}(U \cap U')$.

We note that

$$J(b_\mu) = b'_\mu \text{ and } J^{-1}(b'_\mu) = b_\mu. \quad (220)$$

Take $a \in \mathcal{V}(U \cap U')$, the a -DCDO associated with $\langle U, B \rangle$ and $\langle U', B' \rangle$, namely ∂_a and ∂'_a , are related by

$$\partial'_a v = J(\partial_a J^{-1}(v)). \quad (221)$$

Proof. A straightforward calculation, yields

$$\partial'_a v = (a\beta^{\mu'}(v))b'_\mu = (a\beta^{\mu'}(v))J(b_\mu).$$

Using the identity $\beta^\mu(J^{-1}(v)) = \beta^{\mu'}(v)$ (valid for smooth extensor fields) we get

$$\partial'_a v = J((a\beta^\mu(J^{-1}(v)))b_\mu),$$

from where the expected result follows. ■

We see that from the definition of deformed parallelism structure, ∂'_a is a *J-deformation* of ∂_a .

Then, from Eq.(136), we have

$$\partial'_a \omega = J^{-\Delta}(\partial_a J^{\Delta}(\omega)). \quad (222)$$

Finally, we note that

$$J^{-\Delta}(\beta^\mu) = \beta^{\mu'} \text{ and } J^{\Delta}(\beta^{\mu'}) = \beta^\mu. \quad (223)$$

References

- [1] Choquet-Bruhat, Y., de Witt-Morette, C. and Dillard-Bleick, M., *Analysis, Manifolds and Physics*, North-Holland Publ. Co., Amsterdam, 1982.
- [2] Crumeyrolle, A., *Orthogonal and Symplectic Clifford Algebras — Spinor Structures* (Kluwer Acad. Publ, Dordrecht, 1990).
- [3] Eddington, A. S., *The Mathematical Theory of Relativity* (3rd edn.), Chelsea, New York, 1975.
- [4] Fernández, V. V., Moya, A. M., Notte-Cuello, E., and Rodrigues, W. A. Jr., Duality Products of Multivectors and Multiforms and Extensors, *Algebras, Groups and Geometries* **24** (1), 24-54 (2007).
- [5] Fernández, V. V., Moya, A. M., Notte-Cuello, E., and Rodrigues, W. A. Jr., Parallelism Structure on a Smooth Manifold, *Algebras, Groups and Geometries* **24** (2), 129-155 (2007).
- [6] Fernández, V. V., Moya, A. M., Notte-Cuello, E., and Rodrigues, W. A. Jr., Covariant Differentiation of Multivector and Multiform Fields, *Algebras, Groups and Geometries* **24** (2), 221-237 (2007).
- [7] Fernández, V. V., Moya, A. M., Notte-Cuello, E., and Rodrigues, W. A. Jr., Covariant Differentiation of Extensor Fields, *Algebras, Groups and Geometries* **24** (3), 255-267 (2007).
- [8] Fernández, V. V., Moya, A. M., and Rodrigues, W. A. Jr., Geometric Algebras and Extensors, *Int. J. Geom. Meth. Math. Phys* **4** (6) 927-964 (2007).
- [9] Fernández, V. V., Moya, A. M., and Rodrigues, W. A. Jr., Applications of Geometric and Extensor Algebras in the Study of the Differential Geometry of Arbitrary Manifolds, *Int. J. Geom. Meth. Math. Phys.* **4** (7), 1117-1158 (2007).
- [10] Fernández, V. V., Moya, A. M., da Rocha, R., and Rodrigues, W. A. Jr., Riemann and Ricci Fields in Geometric Structures, *Int. J. Geom. Meth. Math. Phys.* **4** (7), 1159-1172 (2007).

- [11] Fernández, V. V., Moya, A. M., and Rodrigues, W. A., Jr., Multivector and Extensor Calculus, Special Issue of Adv. in Appl. Clifford Algebras 11(S3), 1-103 (2001).
- [12] Fernández, V. V., and Rodrigues, W. A. Jr., *Gravitation as a Plastic Distortion of the Lorentz Vacuum*, Fundamental Theories of Physics **168**, Springer, Berlin, 2010.
- [13] Geroch, R. "Spin structure of space-times in general relativity I," J. Math. Phys. 9, 1739–1744 (1968).
- [14] Geroch, R. "Spin structure of space-times in general relativity II," J. Math. Phys. 11, 343–348 (1970).
- [15] Greub, W., *Multilinear Algebra*, Springer-Verlag, New York, 1967.
- [16] Hestenes, D., and Sobczyk, G., Clifford Algebras to Geometrical Calculus, D. Reidel Publ. Co., Dordrecht, 1984.
- [17] Knus, M., *Quadratic Forms, Clifford Algebras and Spinors*, Seminars in Mathematics, IMECC–UNICAMP, Campinas, 1988.
- [18] Lasenby, A., Doran, C., and Gull, S., Gravity, Gauge Theories and Geometric Algebras, Phil. Trans. R. Soc. 356, 487-582 (1998).
- [19] Lounesto, P., Clifford Algebras and Hestenes Spinors, *Found. Phys.* **23** 1203–1237, (1993).
- [20] Milnor, J. "Spin structures on manifolds," L'Enseignement Mathématique 9, 198–203 (1963).
- [21] Mosna, R. A. and Rodrigues, W. A., Jr., The Bundles of Algebraic and Dirac-Hestenes Spinor Fields, *J. Math. Phys.* **45**, 2945-2966 (2004). [arXiv.org/math-ph/0212033]
- [22] Moya, A. M., Fernández, V. V., and Rodrigues, W. A. Jr., Multivector and Extensor Fields in Smooth Manifolds, *Int. J. Geom. Meth. Math. Phys.* **4** (6) 965-985 (2007).
- [23] da Rocha R., and Vaz, J., Jr., Extended Grassmann and Clifford algebras, *Adv. Appl. Clifford Algebras* **16**, 103-125 (2006). [arXiv.org/math-ph/0603050]
- [24] Rodrigues, W. A. Jr. and Souza, Q. A. G., Spinor Fields and Superfields as Equivalence Classes of Exterior Algebra Fields, in Lounesto, P. and Ablamowicz, R. (eds.), *Clifford Algebras and Spinor Structures*, Mathematics and its Applications **321**, 177-198, Kluwer Acad. Publ., Dordrecht, 1995.
- [25] Rodrigues, W. A. Jr., Algebraic and Dirac-Hestenes Spinors and Spinor Fields, *J. Math. Physics* **45**, 2908-2944 (2004). [arXiv.org/math-ph/0212030]

- [26] Rodrigues, W. A. Jr. and Souza, Q. A. G., The Hyperbolic Clifford Algebra of Multivectors, *Algebras, Groups and Geometries* **24** (1), 1-24 (2007).
- [27] Rodrigues, W. A. Jr., and Oliveira, E. Capelas, *The Many Faces of Maxwell, Dirac and Einstein Equations. A Clifford Bundle Approach*, Lecture Notes in Physics **722**, Springer, New York, 2007.
- [28] Rodrigues, W. A. Jr. , and Wainer, S. A., *A Clifford Bundle Approach to the Differential Geometry of Branes, 2013* [[arXiv:1309.4007v2](#) [math-ph]] .
- [29] Sachs, R. K., and Wu, H., General Relativity for Mathematicians, Springer-Verlag, New York 1977.
- [30] Witten, E., A Note on the Antibracket Formalism, *Mod. Phys. Lett. A* **5**, 487-494 (1990).